Differentiation

Matthew Williams • Add Math • May 16, 2026

Differentiation

Differentiation answers one question: how does a quantity change as another quantity changes? The answer is the **derivative**, a function that gives the instantaneous rate of change at any point. Calculus accounts for five Paper 01 items and two Paper 02 questions.

The Derivative as a Gradient

For a straight line, the gradient is constant. For a curve, the gradient varies from point to point. The derivative $\frac{dy}{dx}$ (also written $f'(x)$) gives the gradient of the tangent to $y = f(x)$ at any x value.

The derivative is defined as a limit:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is the slope of the chord between $(x, f(x))$ and $(x+h, f(x+h))$ as h approaches zero. The syllabus expects an intuitive understanding of this limit, not formal proofs.

The Power Rule

The most-used differentiation rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Works for any real n, including fractions and negative values.

Function	Derivative
[Math: x^5]	[Math: 5x^4]

[Math: x^{-3}]	[Math: -3x^{-4} = -\frac{3}{x^4}]
[Math: x^{1/2} = \sqrt{x}]	[Math: \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}]
constant [Math: c]	[Math: 0]

Linear rules: $\frac{d}{dx}[cf(x)$ = cf'(x)] and $\frac{d}{dx}[f(x) + g(x)$ = f'(x) + g'(x)].

Differentiate polynomial functions term by term.

📋 **Example**

Differentiate $y = 4x^3 - \frac{2}{x} + 5\sqrt{x}$.

Rewrite as $y = 4x^3 - 2x^{-1} + 5x^{1/2}$, then differentiate:

$$\frac{dy}{dx} = 12x^2 + 2x^{-2} + \frac{5}{2}x^{-1/2} = 12x^2 + \frac{2}{x^2} + \frac{5}{2\sqrt{x}}$$

Derivatives of Sine and Cosine

$$\frac{d}{dx}(\sin ax) = a\cos ax \qquad \frac{d}{dx}(\cos ax) = -a\sin ax$$

The chain applied to a linear argument ax simply multiplies by the inner derivative a.

📋 **Example**

$\frac{d}{dx}(\sin 3x) = 3\cos 3x$, $\quad \frac{d}{dx}(\cos 5x) = -5\sin 5x$

The Chain Rule

For a composite function $y = f(g(x))$, written as $y = f(u)$ where $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Informally: differentiate the outer function (leaving the inner unchanged), then multiply by the derivative of the inner function.

📋 **Example**

Differentiate $y = (3x^2 + 1)^5$.

Let $u = 3x^2 + 1$, so $y = u^5$.

$$\frac{dy}{du} = 5u^4, \qquad \frac{du}{dx} = 6x$$

$$\frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot 6x = 30x(3x^2 + 1)^4$$

📋 **Example**

Differentiate $y = \sin(x^2 + 1)$.

Outer: $\sin(\cdot)$ differentiates to $\cos(\cdot)$. Inner: $x^2 + 1$ differentiates to $2x$.

$$\frac{dy}{dx} = 2x\cos(x^2 + 1)$$

The Product Rule

For $y = u(x)\,v(x)$:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

"First times derivative of second, plus second times derivative of first."

📋 **Example**

Differentiate $y = x^2 \sin 3x$.

$u = x^2$, $v = \sin 3x$, $u' = 2x$, $v' = 3\cos 3x$.

$$\frac{dy}{dx} = x^2(3\cos 3x) + \sin 3x(2x) = 3x^2\cos 3x + 2x\sin 3x$$

The Quotient Rule

For $y = \frac{u(x)}{v(x)}$:

$$\frac{dy}{dx} = \frac{v\,u' - u\,v'}{v^2}$$

"Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared."

📋 **Example**

Differentiate $y = \frac{x^2+1}{2x-3}$.

$u = x^2 + 1$, $v = 2x - 3$, $u' = 2x$, $v' = 2$.

$$\frac{dy}{dx} = \frac{(2x-3)(2x) - (x^2+1)(2)}{(2x-3)^2} = \frac{4x^2 - 6x - 2x^2 - 2}{(2x-3)^2} = \frac{2x^2 - 6x - 2}{(2x-3)^2}$$

Tangents and Normals to Curves

- The **tangent** at $x = a$ has gradient $f'(a)$.
- The **normal** at $x = a$ is perpendicular to the tangent; its gradient is $-1/f'(a)$.

Both use the point-slope form $y - y0 = m(x - a)$ where $(a, y0) = (a, f(a))$.

📋 **Example**

Find equations of the tangent and normal to $y = x^3 - 2x + 1$ at $x = 2$.

$y(2) = 8 - 4 + 1 = 5$. Point: $(2, 5)$.

$y' = 3x^2 - 2$, so $y'(2) = 12 - 2 = 10$. Tangent gradient: 10 .

Tangent: $y - 5 = 10(x - 2) \Rightarrow y = 10x - 15$

Normal gradient: $-\frac{1}{10}$. Normal: $y - 5 = -\frac{1}{10}(x - 2) \Rightarrow y = -\frac{x}{10} + \frac{27}{5}$

Stationary Points

A stationary point occurs where $\frac{dy}{dx} = 0$. At these points the tangent is horizontal.

Second derivative test:

- If $\frac{d^2y}{dx^2} > 0$ at a stationary point: **minimum** (curve concave up).
- If $\frac{d^2y}{dx^2} < 0$ at a stationary point: **maximum** (curve concave down).
- If $\frac{d^2y}{dx^2} = 0$: test is inconclusive — use the **first derivative sign-change test** instead.

First derivative sign-change test: evaluate $\frac{dy}{dx}$ at values just before and just after the stationary point.

- $+ \to -$: maximum (gradient goes from positive to negative).
- $- \to +$: minimum (gradient goes from negative to positive).
- Same sign on both sides: point of inflection (not a turning point).

> 📋 **Example**
> Find and classify the stationary points of $y = x^3 - 3x^2 - 9x + 2$.
>
> $\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1) = 0$
>
> Stationary points at $x = 3$ and $x = -1$.
>
> $\frac{d^2y}{dx^2} = 6x - 6$
>
> At $x = 3$: $\frac{d^2y}{dx^2} = 12 > 0$: minimum. $y = 27 - 27 - 27 + 2 = -25$. Point: $(3, -25)$.
>
> At $x = -1$: $\frac{d^2y}{dx^2} = -12 < 0$: maximum. $y = -1 - 3 + 9 + 2 = 7$. Point: $(-1, 7)$.

Rates of Change and Kinematics

The derivative represents any rate of change, not just geometric gradient.

For a particle with **displacement** $s(t)$:

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

- When $v = 0$: particle is instantaneously at rest.

- When $a = 0$: velocity is at a stationary value (often maximum velocity).
- Positive v : moving in the positive direction; negative v

: moving in the negative direction.

📋 **Example**

A particle moves so that its displacement at time t seconds is $s = t^3 - 6t^2 + 9t$.

Find the velocity and acceleration, and determine when the particle is at rest.

$v = 3t^2 - 12t + 9 = 3(t-1)(t-3)$. At rest when $v = 0$: $t = 1$ or $t = 3$.

$a = 6t - 12$. At $t = 1$: $a = -6$ (decelerating). At $t = 3$: $a = 6$ (accelerating).

Connected Rates of Change

When two quantities both change with time, the chain rule links their rates:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

📝 **Exam Tip**

In connected-rates questions, the chain rule always links $\frac{d(\text{target})}{dt}$ through the connecting quantity. Write out the chain rule expression first before substituting values.

📋 **Example**

A spherical balloon is being inflated so that its radius increases at 0.5 cm/s. Find the rate at which its volume is increasing when the radius is 3 cm.

Volume of sphere: $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$.

Given: $\frac{dr}{dt} = 0.5$. Want: $\frac{dV}{dt}$.

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \times 0.5 = 2\pi r^2$$

At $r = 3$: $\frac{dV}{dt} = 2\pi(9) = 18\pi$ cm^3/s.

Applied Optimisation

Differentiation finds the value of a variable that maximises or minimises a given quantity. The standard exam scenario involves a profit or revenue function.

For any business model:

- **Revenue** $R(x)$: income from selling x units.
- **Cost** $C(x)$: total cost to produce x units (fixed costs plus variable costs).
- **Profit** $P(x) = R(x) - C(x)$.

To maximise profit: set $P'(x) = 0$, solve for x, then verify with the second derivative.

📋 **Example**

A company's revenue and cost functions are $R(x) = 196x - 3x^2$ and $C(x) = 14 + 4x$.

Profit function:

$$P(x) = (196x - 3x^2) - (14 + 4x) = -3x^2 + 192x - 14$$

Maximise: set $P'(x) = 0$.

$$P'(x) = -6x + 192 = 0 \Rightarrow x = 32$$

Maximum profit: $P(32) = -3(32)^2 + 192(32) - 14 = 3058$ (hundreds of dollars).

Verify: $P''(x) = -6 < 0$, confirming a maximum.

Maximum profit of 3058 is achieved when 32 units are sold.