

# Differentiation

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## Differentiation

Differentiation answers one question: how does a quantity change as another quantity changes? The answer is the **derivative**, a function that gives the instantaneous rate of change at any point. Calculus accounts for five Paper 01 items and two Paper 02 questions.

## The Derivative as a Gradient

For a straight line, the gradient is constant. For a curve, the gradient varies from point to point. The derivative

$\frac{dy}{dx}$  (also written  $f'(x)$ ) gives the gradient of the tangent to  $y = f(x)$  at any  $x$  value.

The derivative is defined as a limit:

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

This is the slope of the chord between  $(x, f(x))$  and  $(x+h, f(x+h))$  as  $h$  approaches zero. The syllabus expects an intuitive understanding of this limit, not formal proofs.

## The Power Rule

The most-used differentiation rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Works for any real  $n$ , including fractions and negative values.

Function	Derivative
[Math: x^5]	[Math: 5x^4]

[Math: $x^{-3}$ ]	[Math: $-3x^{-4} = -\frac{3}{x^4}$ ]
[Math: $x^{1/2} = \sqrt{x}$ ]	[Math: $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ ]
constant [Math: $c$ ]	[Math: $0$ ]

**Linear rules:**  $\frac{d}{dx}[cf(x)] = cf'(x)$  and  $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$ .

Differentiate polynomial functions term by term.

### Example

Differentiate  $y = 4x^3 - \frac{2}{x} + 5\sqrt{x}$ .

Rewrite as  $y = 4x^3 - 2x^{-1} + 5x^{1/2}$ , then differentiate:

$$\frac{dy}{dx} = 12x^2 + 2x^{-2} + \frac{5}{2}x^{-1/2} = 12x^2 + \frac{2}{x^2} + \frac{5}{2\sqrt{x}}$$

## Derivatives of Sine and Cosine

$$\frac{d}{dx}(\sin ax) = a \cos ax \quad \frac{d}{dx}(\cos ax) = -a \sin ax$$

The chain applied to a linear argument  $ax$  simply multiplies by the inner derivative  $a$ .

### Example

$$\frac{d}{dx}(\sin 3x) = 3 \cos 3x, \quad \frac{d}{dx}(\cos 5x) = -5 \sin 5x$$

## The Chain Rule

For a composite function  $y = f(g(x))$ , written as  $y = f(u)$  where  $u = g(x)$ :

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Informally: differentiate the outer function (leaving the inner unchanged), then multiply by the derivative of the inner function.

**Example**

Differentiate  $y = (3x^2 + 1)^5$ .

Let  $u = 3x^2 + 1$ , so  $y = u^5$ .

$$\frac{dy}{du} = 5u^4, \quad \frac{du}{dx} = 6x$$

$$\frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot 6x = 30x(3x^2 + 1)^4$$

**Example**

Differentiate  $y = \sin(x^2 + 1)$ .

Outer:  $\sin(\cdot)$  differentiates to  $\cos(\cdot)$ . Inner:  $x^2 + 1$  differentiates to  $2x$ .

$$\frac{dy}{dx} = 2x \cos(x^2 + 1)$$

## The Product Rule

For  $y = u(x)v(x)$ :

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

"First times derivative of second, plus second times derivative of first."

**Example**

Differentiate  $y = x^2 \sin 3x$ .

$u = x^2$ ,  $v = \sin 3x$ ,  $u' = 2x$ ,  $v' = 3 \cos 3x$ .

$$\frac{dy}{dx} = x^2(3 \cos 3x) + \sin 3x(2x) = 3x^2 \cos 3x + 2x \sin 3x$$

## The Quotient Rule

For  $y = \frac{u(x)}{v(x)}$ :

$$\frac{dy}{dx} = \frac{v u' - u v'}{v^2}$$

"Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared."

### Example

Differentiate  $y = \frac{x^2 + 1}{2x - 3}$ .

$$u = x^2 + 1, v = 2x - 3, u' = 2x, v' = 2.$$

$$\frac{dy}{dx} = \frac{(2x - 3)(2x) - (x^2 + 1)(2)}{(2x - 3)^2} = \frac{4x^2 - 6x - 2x^2 - 2}{(2x - 3)^2} = \frac{2x^2 - 6x - 2}{(2x - 3)^2}$$

## Tangents and Normals to Curves

- The **tangent** at  $x = a$  has gradient  $f'(a)$ .
- The **normal** at  $x = a$  is perpendicular to the tangent; its gradient is  $-1/f'(a)$ .
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Both use the point-slope form  $y - y_0 = m(x - a)$  where  $(a, y_0) = (a, f(a))$ .

### Example

Find equations of the tangent and normal to  $y = x^3 - 2x + 1$  at  $x = 2$ .

$$y(2) = 8 - 4 + 1 = 5. \text{ Point: } (2, 5).$$

$$y' = 3x^2 - 2, \text{ so } y'(2) = 12 - 2 = 10. \text{ Tangent gradient: } 10.$$

$$\text{Tangent: } y - 5 = 10(x - 2) \Rightarrow y = 10x - 15$$

$$\text{Normal gradient: } -\frac{1}{10}. \text{ Normal: } y - 5 = -\frac{1}{10}(x - 2) \Rightarrow y = -\frac{x}{10} + \frac{27}{5}$$

## Stationary Points

A stationary point occurs where  $\frac{dy}{dx} = 0$ . At these points the tangent is horizontal.

### Second derivative test:

- If  $\frac{d^2y}{dx^2} > 0$  at a stationary point: **minimum** (curve concave up).
- If  $\frac{d^2y}{dx^2} < 0$  at a stationary point: **maximum** (curve concave down).
- If  $\frac{d^2y}{dx^2} = 0$ : test is inconclusive — use the **first derivative sign-change test** instead.

**First derivative sign-change test:** evaluate  $\frac{dy}{dx}$

at values just before and just after the stationary point.

- $+$   $\rightarrow$   $-$  : maximum (gradient goes from positive to negative).
- $-$   $\rightarrow$   $+$  : minimum (gradient goes from negative to positive).
- Same sign on both sides: point of inflection (not a turning point).

### Example

Find and classify the stationary points of  $y = x^3 - 3x^2 - 9x + 2$ .

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0$$

Stationary points at  $x = 3$  and  $x = -1$ .

$$\frac{d^2y}{dx^2} = 6x - 6$$

At  $x = 3$ :  $\frac{d^2y}{dx^2} = 12 > 0$ : minimum.  $y = 27 - 27 - 27 + 2 = -25$ . Point:  $(3, -25)$ .

At  $x = -1$ :  $\frac{d^2y}{dx^2} = -12 < 0$ : maximum.  $y = -1 - 3 + 9 + 2 = 7$ . Point:  $(-1, 7)$ .

## Rates of Change and Kinematics

The derivative represents any rate of change, not just geometric gradient.

For a particle with **displacement**  $s(t)$ :

$$v = \frac{ds}{dt} \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

- When  $v = 0$ : particle is instantaneously at rest.

- When  $a = 0$  : velocity is at a stationary value (often maximum velocity).
- Positive  $v$  : moving in the positive direction; negative  $v$  : moving in the negative direction.

### Example

A particle moves so that its displacement at time  $t$  seconds is  $s = t^3 - 6t^2 + 9t$ .

Find the velocity and acceleration, and determine when the particle is at rest.

$v = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$ . At rest when  $v = 0$ :  $t = 1$  or  $t = 3$ .

$a = 6t - 12$ . At  $t = 1$ :  $a = -6$  (decelerating). At  $t = 3$ :  $a = 6$  (accelerating).

## Connected Rates of Change

When two quantities both change with time, the chain rule links their rates:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

### Exam Tip

In connected-rates questions, the chain rule always links  $\frac{d(\text{target})}{dt}$  through the connecting quantity. Write out the chain rule expression first before substituting values.

### Example

A spherical balloon is being inflated so that its radius increases at  $0.5$  cm/s. Find the rate at which its volume is increasing when the radius is  $3$  cm.

Volume of sphere:  $V = \frac{4}{3}\pi r^3$ , so  $\frac{dV}{dr} = 4\pi r^2$ .

Given:  $\frac{dr}{dt} = 0.5$ . Want:  $\frac{dV}{dt}$ .

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \times 0.5 = 2\pi r^2$$

At  $r = 3$ :  $\frac{dV}{dt} = 2\pi(9) = 18\pi$  cm<sup>3</sup>/s.

## Applied Optimisation

Differentiation finds the value of a variable that maximises or minimises a given quantity. The standard exam scenario involves a profit or revenue function.

For any business model:

- **Revenue  $R(x)$**  : income from selling  $x$  units.
- **Cost  $C(x)$**  : total cost to produce  $x$  units (fixed costs plus variable costs).
- **Profit  $P(x) = R(x) - C(x)$**  .

To maximise profit: set  $P'(x) = 0$ , solve for  $x$ , then verify with the second derivative.

### Example

A company's revenue and cost functions are  $R(x) = 196x - 3x^2$  and  $C(x) = 14 + 4x$ .

**Profit function:**

$$P(x) = (196x - 3x^2) - (14 + 4x) = -3x^2 + 192x - 14$$

**Maximise:** set  $P'(x) = 0$ .

$$P'(x) = -6x + 192 = 0 \Rightarrow x = 32$$

**Maximum profit:**  $P(32) = -3(32)^2 + 192(32) - 14 = 3058$  (hundreds of dollars).

**Verify:**  $P''(x) = -6 < 0$ , confirming a maximum.

Maximum profit of **3058** is achieved when **32** units are sold.