Integration

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Integration

Integration is the reverse of differentiation. Given a rate of change, integration recovers the original quantity. It also computes area under curves and volume of revolution. Integration accounts for five Paper 01 items and contributes substantially to Paper 02 calculus questions.

Integration as the Reverse of Differentiation

If $\frac{d}{dx}[F(x)] = f(x)$, then $F(x)$ is an **antiderivative** of $f(x)$.

Since the derivative of any constant is zero, adding any constant to $F(x)$ still gives $f(x)$ on differentiation. This is why indefinite integrals include $+C$:

$$\int f(x)\,dx = F(x) + C$$

> 📋 **Example**
> Since $\frac{d}{dx}(x^3) = 3x^2$, it follows that $\int 3x^2\,dx = x^3 + C$.

The Power Rule for Integration

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

Integration increases the power by 1 and divides by the new power. This is the reverse of the differentiation power rule.

Linear rules:

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Integrate polynomial functions term by term.

📋 **Example**

Find $\int (6x^2 - 4x + 5)\,dx$.

$$= \frac{6x^3}{3} - \frac{4x^2}{2} + 5x + C = 2x^3 - 2x^2 + 5x + C$$

📋 **Example**

Find $\int (x^{-3} + \sqrt{x})dx$.

$$= \frac{x^{-2}}{-2} + \frac{x^{3/2}}{3/2} + C = -\frac{1}{2x^2} + \frac{2}{3}x^{3/2} + C$$

Integrating [Math: (ax + b)^n]

For a linear inner function:

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \qquad (n \neq -1)$$

The a in the denominator comes from dividing by the inner derivative.

📋 **Example**

$\int (3x-1)^4\,dx = \frac{(3x-1)^5}{3 \cdot 5} + C = \frac{(3x-1)^5}{15} + C$

Integrating Trigonometric Functions

Since $\frac{d}{dx}(\sin ax) = a\cos ax$, reversing gives:

$$\int \cos ax\,dx = \frac{\sin ax}{a} + C$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a} + C$$

The sign change for $\sin$ (derivative of $\cos$ is $-\sin$) must be remembered.

📋 **Example**

$\int \sin 4x\,dx = -\frac{\cos 4x}{4} + C$

$\int 3\cos 2x\,dx = \frac{3\sin 2x}{2} + C$

🧠 **Remember**

There is no product or quotient rule for integration at CSEC level. When the integrand is a product or a quotient, expand or simplify it into separate terms first, then integrate term by term.

Finding a Curve from Its Gradient

Given $\frac{dy}{dx} = f(x)$ and a point on the curve, integrate to find y, then use the point to determine C.

📋 **Example**

A curve has gradient $\frac{dy}{dx} = 3x^2 - 4x + 1$ and passes through $(2, 5)$. Find y.

$$y = \int (3x^2 - 4x + 1)\,dx = x^3 - 2x^2 + x + C$$

Substitute $(2, 5)$: $5 = 8 - 8 + 2 + C \Rightarrow C = 3$.

$$y = x^3 - 2x^2 + x + 3$$

Finding a Curve from the Second Derivative

When given $\frac{d^2y}{dx^2}$
, integrate twice. Each integration introduces a new constant, so two separate conditions are needed to determine both.

📋 **Example**

Given $\frac{d^2y}{dx^2} = 12x$, and that $\frac{dy}{dx} = 4$ when $x = 1$, find y given that $y = 7$ when $x = 1$.

First integration:

$$\frac{dy}{dx} = \int 12x \, dx = 6x^2 + A$$

Using $\frac{dy}{dx} = 4$ when $x = 1$: $4 = 6 + A \Rightarrow A = -2$, so $\frac{dy}{dx} = 6x^2 - 2$.

Second integration:

$$y = \int (6x^2 - 2) \, dx = 2x^3 - 2x + B$$

Using $y = 7$ when $x = 1$: $7 = 2 - 2 + B \Rightarrow B = 7$.

$$y = 2x^3 - 2x + 7$$

Definite Integrals

A definite integral has limits and produces a numerical value:

$$\int_a^b f(x) \, dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

No $+C$ is needed in definite integrals (it cancels when subtracting).

📋 **Example**

Evaluate $\int_1^4 (2x - 3) \, dx$.

$$\Big[x^2 - 3x\Big]_1^4 = (16 - 12) - (1 - 3) = 4 - (-2) = 6$$

Area Under a Curve

The definite integral $\int_a^b f(x)\,dx$ gives the **signed** area between the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

When the curve lies **above** the x-axis, the integral is positive and equals the geometric area.

When the curve lies **below** the x-axis, the integral is negative. To find the actual geometric area, take the absolute value.

The syllabus restricts area questions to regions in the **first quadrant**, so the curve will be above the x-axis in the required region.

📋 **Example**

Find the area bounded by $y = x^2 + 1$, the x-axis, $x = 0$, and $x = 3$.

$$\int_0^3 (x^2 + 1)\,dx = \left[\frac{x^3}{3} + x\right]_0^3 = (9 + 3) - 0 = 12$$

Area $= 12$ square units.

Area Between Two Curves

When one curve $f(x)$ lies above another $g(x)$ on $[a, b]$:

$$\int_a^b [f(x) - g(x)]\,dx$$

Find the limits of integration by solving $f(x) = g(x)$.

📋 **Example**

Find the area enclosed by $y = 4 - x^2$ and $y = x + 2$.

Find intersections: $4 - x^2 = x + 2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$, so $x = -2$ and $x = 1$.

$$\int_{-2}^{1} [(4 - x^2) - (x + 2)]\, dx = \int_{-2}^{1} (2 - x - x^2)\, dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \frac{9}{2}$$

Area $= \frac{9}{2}$ sq units.

Volume of Revolution About the x-axis

When the region between $y = f(x)$, the x-axis, and the lines $x = a$, $x = b$ is rotated 360° about the x-axis, the volume of the solid formed is:

$$V = \pi \int_a^b [f(x)]^2\, dx$$

The syllabus restricts this to polynomials of degree at most 2, with the region in the first quadrant.

📋 **Example**

The region bounded by $y = x + 1$, the x-axis, $x = 0$, and $x = 2$ is rotated about the x-axis. Find the volume.

$$V = \pi \int_0^2 (x + 1)^2\, dx = \pi \int_0^2 (x^2 + 2x + 1)\, dx$$

$$= \pi \left[\frac{x^3}{3} + x^2 + x\right]_0^2 = \pi \left(\frac{8}{3} + 4 + 2\right) = \frac{26\pi}{3}$$

Volume $= \frac{26\pi}{3}$ cubic units.

📝 **Exam Tip**

The volume formula uses $[f(x)]^2$, not $[f(x)]$
. Square the function before integrating. Also, the factor of π sits outside the integral.

Kinematics and Integration

Integration connects acceleration, velocity, and displacement:

$$v = \int a\,dt + C_1 \qquad s = \int v\,dt + C_2$$

Use given initial conditions (values at $t = 0$ or another specified time) to find the constants.

📋 **Example**

A particle starts from rest at the origin. Its acceleration is $a = 6t - 4$.

Find velocity: $v = \int (6t-4)\,dt = 3t^2 - 4t + C$. At $t = 0$, $v = 0$, so $C = 0$. Thus $v = 3t^2 - 4t$.

Find displacement: $s = \int (3t^2 - 4t)\,dt = t^3 - 2t^2 + K$. At $t = 0$, $s = 0$, so $K = 0$. Thus $s = t^3 - 2t^2$.

🧠 **Remember**

"Starts from rest" means $v = 0$ at $t = 0$
. "Starts from the origin" or "starts at a fixed point" means $s = 0$ at $t = 0$
. These initial conditions determine the constants of integration.