Kinematics

Matthew Williams • Add Math • May 16, 2026

Kinematics

Kinematics is the study of how objects move, without asking why they move. At CSEC Additional Mathematics level the central idea is that displacement, velocity, and acceleration are not three separate things but one quantity viewed at different levels: each is obtained from the previous one by differentiation, and each is recovered from the next one by integration.

The Three Kinematic Quantities

A particle moves along a straight line. Its position is measured from a fixed reference point O, and the measurement is called

displacement, usually written x or s .

- Displacement can be positive (one side of O) or negative (the other side). It is a vector.
- **Velocity** v is the rate at which displacement changes: $v = \frac{dx}{dt}$.
- **Acceleration** a is the rate at which velocity changes: $a = \frac{dv}{dt}$.

Because velocity is itself already a derivative of displacement, acceleration can also be written as the second derivative:

$$a = \frac{d^2x}{dt^2}$$

The formula sheet uses the dot notation $\dot{x} = v$ and $\ddot{x} = a$

, where each dot represents one differentiation with respect to time.

🧠 **Remember**

The direction of each quantity matters. A negative velocity means the particle is moving in the opposite direction to the chosen positive direction. A negative acceleration means velocity is decreasing (not necessarily that the particle is slowing down: if the particle is already moving in the negative direction, negative acceleration speeds it up).

Equations of Motion Under Constant Acceleration

When acceleration is constant throughout the motion, five quantities describe it: initial velocity u, final velocity v, displacement s, acceleration a, and time t. The four **equations of motion** (commonly called SUVAT) relate these in pairs:

Equation	Quantity not involved
[Math: v = u + at]	[Math: s]
[Math: s = ut + \dfrac{1}{2}at^2]	[Math: v]
[Math: v^2 = u^2 + 2as]	[Math: t]
[Math: s = \dfrac{1}{2}(u + v)t]	[Math: a]

Choosing the right equation: list the three quantities you know and identify the one you want. The fourth column shows which quantity each equation omits, so choose the equation that does not involve the quantity that is neither given nor needed.

📝 **Exam Tip**

The SUVAT equations are only valid when acceleration is constant throughout the motion. If the question gives displacement or velocity as a polynomial in t, acceleration is variable and you must use calculus.

📋 **Example**

A particle starts from rest and accelerates uniformly at 3 m/s² along a straight line. Find its velocity after 5 s and the distance covered.

Known: $u = 0$, $a = 3$, $t = 5$.

For v: use $v = u + at = 0 + 3(5) = 15$ m/s.

For s: use $s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}(3)(25) = 37.5$ m.

Deriving the Equations Using Calculus

The SUVAT equations are not independent formulas; they are what you get when you integrate constant acceleration. Starting from $a =$ constant:

Integrate a with respect to t:

$$v = \int a \, dt = at + C$$

At $t = 0$, $v = u$, so $C = u$. This gives $v = u + at$.

Integrate v with respect to t:

$$s = \int (u + at) \, dt = ut + \frac{1}{2}at^2 + K$$

Measuring displacement from the starting position ($s = 0$ at $t = 0$) gives $K = 0$, so $s = ut + \frac{1}{2}at^2$.

The other two equations follow algebraically by eliminating t or a

. This connection between calculus and the equations of motion is what the syllabus calls "calculus verification of kinematics equations."

Finding Velocity and Acceleration by Differentiation

If displacement is given as a function of time, differentiate once to get velocity, and again to get acceleration.

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

📋 **Example**

A particle moves along a straight line so that its displacement from a fixed point O is

$$x = t^3 - 9t^2 + 24t \quad (t \geq 0)$$

where x is in metres and t in seconds.

Find the velocity and acceleration, and determine when the particle is momentarily at rest.

Step 1: Velocity

$$v = \frac{dx}{dt} = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8) = 3(t-2)(t-4)$$

Step 2: Acceleration

$$a = \frac{dv}{dt} = 6t - 18$$

**Step 3: Momentarily at rest means $v = 0$ **

$$3(t-2)(t-4) = 0$$

so $t = 2$ or $t = 4$.

At $t = 2$: $a = 12 - 18 = -6$ m/s² (particle is decelerating as it pauses).

At $t = 4$: $a = 24 - 18 = 6$ m/s² (particle is accelerating away in the opposite direction).

Stationary Particles and Direction of Motion

A particle is **momentarily at rest**whenever $v = 0$

. To find those moments, form the velocity function and solve $v = 0$.

A particle **changes direction**when $v = 0$ and the sign of v actually switches. Setting $v = 0$ identifies candidate moments; checking the sign of v

just before and just after confirms whether a reversal really occurs.

📝 **Exam Tip**

In a typical Paper 02 question you may be asked separately: "when is the particle at rest" and "when does the particle change direction." These are not always the same thing. If v touches zero but returns to the same sign, the particle pauses without reversing. Check the sign change.

Speed is the magnitude of velocity: $\text{speed} = |v|$
. Speed is always non-negative, whereas velocity can be negative. When a question asks for speed, give
$|v|$, not v.

Initial Conditions: Reading Displacement at [Math: t = 0]

Many questions state the initial displacement of the particle. Substituting $t = 0$
into the displacement function recovers this directly. A particle "at the origin" at $t = 0$ means $x = 0$ when $t = 0$; a particle starting "8 m from O" means $x = 8$ when $t = 0$.

This matters most in integration problems, where the constant of integration must be pinned down by an initial condition.

Recovering Velocity and Displacement by Integration

When the question gives acceleration and asks for velocity, or gives velocity and asks for displacement, integrate.

$$v = \int a\,dt + C_1 \qquad x = \int v\,dt + C_2$$

Each integration produces a constant. To find the constant, substitute the known value of the quantity at a specific time (usually
$t = 0$).

Common initial condition phrases:

Phrase	What it means mathematically
"starts from rest"	[Math: v = 0] when [Math: t = 0]
"passes through O at [Math: t = 0]"	[Math: x = 0] when [Math: t = 0]
"initially at rest at O"	both [Math: v = 0] and [Math: x = 0] when [Math: t = 0]
"starts from a point 5 m from O"	[Math: x = 5] when [Math: t = 0]

📋 **Example**

A particle moves in a straight line. Its acceleration at time t seconds is $a = 6t - 4$. Initially the particle is at rest at a point 3 m from O.

Find the velocity and displacement in terms of t.

Step 1: Integrate acceleration to find velocity

$$v = \int (6t - 4)\, dt = 3t^2 - 4t + C_1$$

"At rest" when $t = 0$ means $v = 0$: substituting gives $0 = 0 - 0 + C1$, so $C1 = 0$.

$$v = 3t^2 - 4t$$

Step 2: Integrate velocity to find displacement

$$x = \int (3t^2 - 4t)\, dt = t^3 - 2t^2 + C_2$$

"3 m from O" when $t = 0$ means $x = 3$: substituting gives $3 = 0 - 0 + C2$, so $C2 = 3$.

$$x = t^3 - 2t^2 + 3$$

Velocity-Time Graphs

A velocity-time graph plots v against t. Two features are tested directly.

Gradient: the gradient at any point equals the acceleration at that instant ($a = dv/dt$). For a straight-line segment (constant acceleration), the gradient is $\Delta v / \Delta t$. A horizontal segment means zero acceleration; a downward slope means negative acceleration (deceleration in the positive direction).

Area:

the area between the graph and the time axis equals displacement in that interval. For a straight-line segment from u to v over time t, the region is a trapezium:

$$s = \frac{1}{2}(u + v)t$$

which is exactly the fourth SUVAT equation. For a curved graph (variable acceleration), the displacement must be found by integration:

$$s = \int v \, dt \, .$$

<KinematicsGraph />

The diagram above illustrates the standard relationships between displacement, velocity, and acceleration as functions of time.

> 📝 **Exam Tip**
> When a v-t graph question asks for acceleration, read the gradient of the relevant segment:
> $a = \Delta v / \Delta t$
> . When it asks for displacement or distance, calculate the area (taking regions below the axis as negative displacement but positive distance).

Interpreting Motion Physically

Before manipulating equations, sketch the motion mentally or on paper:

- Where does the particle start? (Substitute $t = 0$.)
- Does it move left or right initially? (Find the sign of v just after $t = 0$.)
- Where does it pause? (Solve $v = 0$.)
- Does it reverse there? (Check the sign of v on both sides of the root.)
- What is its position at those moments? (Substitute the t values back into x .)

This physical reading of the answer is what examiners look for under the "Reasoning" profile mark.

Exam-Style Worked Example

📋 **Example**

A particle moves in a straight line so that its displacement, x
metres, from a fixed point O at time t seconds is given by

$$x = 2t^3 - 15t^2 + 24t$$

(a) Find the velocity and acceleration of the particle in terms of t.

(b) Find the values of t
when the particle is at instantaneous rest and determine its displacement at those moments.

(c) Find the velocity of the particle when its acceleration is zero.

Part (a):

$$v = \frac{dx}{dt} = 6t^2 - 30t + 24 \qquad a = \frac{dv}{dt} = 12t - 30$$

Part (b): Particle at rest means $v = 0$

$$6t^2 - 30t + 24 = 0$$

Divide by 6:

$$t^2 - 5t + 4 = 0 \implies (t-1)(t-4) = 0$$

so $t = 1$ or $t = 4$.

Displacement at $t = 1$: $x = 2 - 15 + 24 = 11$ m.

Displacement at $t = 4$: $x = 2(64) - 15(16) + 24(4) = 128 - 240 + 96 = -16$ m.

The particle is momentarily at rest at $x = 11$ m (when $t = 1$) and at $x = -16$ m (when $t = 4$).

Part (c): Acceleration zero means $a = 0$

$$12t - 30 = 0 \implies t = 2.5$$

Velocity at $t = 2.5$:

$$v = 6(6.25) - 30(2.5) + 24 = 37.5 - 75 + 24 = -13.5$$

The velocity is -13.5
m/s. The negative sign means the particle is moving in the negative direction at that instant.