Quadratic Functions and Equations

Matthew Williams • Add Math • May 16, 2026

Quadratic Functions and Equations

Quadratics are one of the most examined topics across both papers. Paper 01 includes four dedicated items; Paper 02 frequently builds entire questions around quadratic functions and their graphs.

The General Form

A quadratic function has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. Its graph is a **parabola**. When $a > 0$ the parabola opens upward (minimum point); when $a < 0$ it opens downward (maximum point).

Completing the Square

Completing the square rewrites a quadratic into **vertex form**: $a(x+h)^2 + k$.

The vertex (turning point) of the parabola is $(-h, k)$, and the axis of symmetry is $x = -h$.

Method for $ax^2 + bx + c$:

- 1. Factor out a from the first two terms: $a\left(x^2 + \frac{b}{a}x\right) + c$
- 2. Add and subtract $\left(\frac{b}{2a}\right)^2$ inside the bracket.
- 3. Collect the perfect square and simplify.

📋 **Example**

Express $2x^2 - 8x + 5$ in the form $a(x+h)^2 + k$.

$$2x^2 - 8x + 5 = 2(x^2 - 4x) + 5$$

$$= 2(x^2 - 4x + 4 - 4) + 5$$

$$= 2(x-2)^2 - 8 + 5 = 2(x-2)^2 - 3$$

Vertex: $(2, -3)$. Since $a = 2 > 0$, this is a minimum.

Maximum and Minimum Values

From vertex form $a(x+h)^2 + k$:

- The **minimum value** of $f(x)$ is k, occurring when $x = -h$ (if $a > 0$).
- The **maximum value** of $f(x)$ is k, occurring when $x = -h$ (if $a < 0$).

The **range** follows directly:

- If $a > 0$: range is $f(x) \geq k$.
- If $a < 0$: range is $f(x) \leq k$.

📝 **Exam Tip**

Questions asking for the "range" of a quadratic almost always require completing the square first to identify
k. Quote the range as an inequality: $f(x) \geq k$ or $f(x) \leq k$.

Sketching Quadratic Graphs

A complete sketch of $f(x) = ax^2 + bx + c$ shows:

- The **shape** (upward or downward parabola).
- The **turning point** from completed square form.
- The y**-intercept**: substitute $x = 0$ to get $(0, c)$.
- The x**-intercepts** (if any): solve $ax^2 + bx + c = 0$.
- The **axis of symmetry**: $x = -b/(2a)$.

The diagram below shows all five features for $y = x^2 - 4x + 3$.

<QuadraticSketchDiagram />

The Discriminant and Nature of Roots

For $ax^2 + bx + c = 0$, the **discriminant** is $\Delta = b^2 - 4ac$.

Value of [Math: \Delta]	Nature of roots	Graph interpretation
[Math: \Delta > 0]	Two distinct real roots	Parabola crosses [Math: x]-axis twice
[Math: \Delta = 0]	One repeated (equal) real root	Parabola touches [Math: x]-axis once
[Math: \Delta < 0]	No real roots	Parabola does not meet [Math: x]-axis

📋 **Example**

Determine the values of k for which $kx^2 + (k-3)x + 1 = 0$ has no real roots.

For no real roots: $\Delta < 0$.

$$\Delta = (k-3)^2 - 4k(1) = k^2 - 6k + 9 - 4k = k^2 - 10k + 9$$

Set $\Delta < 0$: $(k-1)(k-9) < 0$, so $1 < k < 9$.

Also, for the equation to be quadratic, $k \neq 0$. Since $k = 0$ is outside the interval $1 < k < 9$, the final answer is $1 < k < 9$.

Methods of Solving Quadratic Equations

Factorisation

is fastest when the quadratic factors cleanly over the integers. Look for two numbers whose product is

ac and sum is b.

Completing the square works for any quadratic and gives the exact roots in surd form.

Quadratic formula: For $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

🧠 **Remember**

When a question says "solve," any valid method is acceptable. When it says "complete the square" or "express in the form $a(x+h)^2+k$," you must use that specific method.

Equations Reducible to a Quadratic

Some equations become quadratic after a substitution.

Common substitutions:

Equation type	Substitution	Resulting quadratic
[Math: x^4 - 5x^2 + 4 = 0]	[Math: u = x^2]	[Math: u^2 - 5u + 4 = 0]
[Math: x - 5\sqrt{x} + 6 = 0]	[Math: u = \sqrt{x}]	[Math: u^2 - 5u + 6 = 0]
[Math: 2^{2x} - 3(2^x) - 4 = 0]	[Math: u = 2^x]	[Math: u^2 - 3u - 4 = 0]

📋 **Example**

Solve $x - 5\sqrt{x} + 6 = 0$.

Let $u = \sqrt{x}$ (so $u \geq 0$ and $x = u^2$):

$$u^2 - 5u + 6 = 0 \Rightarrow (u-2)(u-3) = 0$$

so $u = 2$ or $u = 3$. Since $u = \sqrt{x}$: $x = 4$ or $x = 9$.

Check: $4 - 5(2) + 6 = 0$ and $9 - 5(3) + 6 = 0$

🧠 **Remember**

An **extraneous solution** is a value produced by the algebra that does not satisfy the original equation. It arises because a substitution (like $u = \sqrt{x}$, which requires $u \geq 0$) imposes a domain restriction that the factored quadratic silently ignores. Always verify every back-substituted solution in the original equation. Any value that fails the check must be marked **(Invalid)** and discarded — the algebra will not warn you automatically.

Relationships Between Roots and Coefficients

For $ax^2 + bx + c = 0$ with roots α and β:

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

These allow you to find expressions involving α and β without solving for them individually.

Expression	How to find it
[Math: \alpha^2 + \beta^2]	[Math: (\alpha + \beta)^2 - 2\alpha\beta]
[Math: \alpha^2 \beta + \alpha\beta^2]	[Math: \alpha\beta(\alpha + \beta)]
[Math: \frac{1}{\alpha} + \frac{1}{\beta}]	[Math: \frac{\alpha + \beta}{\alpha\beta}]
[Math: (\alpha - \beta)^2]	[Math: (\alpha + \beta)^2 - 4\alpha\beta]

📋 **Example**

The roots of $3x^2 - 5x + 2 = 0$ are α and β. Find $\alpha^2 + \beta^2$.

$$\alpha + \beta = \frac{5}{3}, \quad \alpha\beta = \frac{2}{3}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{25}{9} - \frac{4}{3} = \frac{25}{9} - \frac{12}{9} = \frac{13}{9}$$

Forming a New Quadratic with Given Roots (Vieta's Root Theorem)

If a question asks for a quadratic whose roots are some transformation of α and β, find the new sum and product, then substitute into:

$$x^2 - (\text{Sum of Roots})x + (\text{Product of Roots}) = 0$$

If the coefficients are not integers, multiply through by the appropriate constant so that $a, b, c \in \mathbb{Z}$.

📋 **Example**

The roots of $2x^2 - 3x + 1 = 0$ are α and β. Find the quadratic with integer coefficients whose roots are α^2 and β^2.

From the original:

$$\alpha + \beta = \frac{3}{2}$$

$$\alpha\beta = \frac{1}{2}$$

New sum:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{9}{4} - 1 = \frac{5}{4}$$

New product:

$$\alpha^2\beta^2 = (\alpha\beta)^2 = \frac{1}{4}$$

Substituting into the formula:

$$x^2 - \frac{5}{4}x + \frac{1}{4} = 0$$

Coefficients are not integers. Multiply through by 4:

$$4x^2 - 5x + 1 = 0$$

Simultaneous Equations: One Linear, One Non-Linear

The standard method is substitution: express one variable from the linear equation, then substitute into the quadratic (or circle) equation.

📋 **Example**

Solve the system: $y = x + 1$ and $x^2 + y^2 = 13$.

Substitute $y = x + 1$ into the circle:

$$x^2 + (x+1)^2 = 13$$

$$x^2 + x^2 + 2x + 1 = 13$$

$$2x^2 + 2x - 12 = 0 \implies x^2 + x - 6 = 0 \implies (x+3)(x-2) = 0$$

$x = -3$ gives $y = -2$; $x = 2$ gives $y = 3$.

Solutions: $(-3, -2)$ and $(2, 3)$.

📝 **Exam Tip**

When a line intersects a circle, the discriminant of the resulting quadratic tells you about the intersection.
$\Delta > 0$: two points; $\Delta = 0$: the line is a tangent; $\Delta < 0$: no intersection.

Study Vault