

Quadratic Functions and Equations

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Quadratic Functions and Equations

Quadratics are one of the most examined topics across both papers. Paper 01 includes four dedicated items; Paper 02 frequently builds entire questions around quadratic functions and their graphs.

The General Form

A quadratic function has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. Its graph is a **parabola**. When $a > 0$ the parabola opens upward (minimum point); when $a < 0$ it opens downward (maximum point).

Completing the Square

Completing the square rewrites a quadratic into **vertex form**: $a(x + h)^2 + k$.

The vertex (turning point) of the parabola is $(-h, k)$, and the axis of symmetry is $x = -h$.

****Method for $ax^2 + bx + c$:****

- 1. Factor out a from the first two terms: $a(x^2 + \frac{b}{a}x) + c$
- 2. Add and subtract $(\frac{b}{2a})^2$ inside the bracket.
- 3. Collect the perfect square and simplify.

Example

Express $2x^2 - 8x + 5$ in the form $a(x + h)^2 + k$.

$$\begin{aligned} 2x^2 - 8x + 5 &= 2(x^2 - 4x) + 5 \\ &= 2(x^2 - 4x + 4 - 4) + 5 \\ &= 2(x - 2)^2 - 8 + 5 = 2(x - 2)^2 - 3 \end{aligned}$$

Vertex: $(2, -3)$. Since $a = 2 > 0$, this is a minimum.

Maximum and Minimum Values

From vertex form $a(x + h)^2 + k$:

- The **minimum value** of $f(x)$ is k , occurring when $x = -h$ (if $a > 0$).
- The **maximum value** of $f(x)$ is k , occurring when $x = -h$ (if $a < 0$).

The **range** follows directly:

- If $a > 0$: range is $f(x) \geq k$.
- If $a < 0$: range is $f(x) \leq k$.

Exam Tip

Questions asking for the "range" of a quadratic almost always require completing the square first to identify

k . Quote the range as an inequality: $f(x) \geq k$ or $f(x) \leq k$.

Sketching Quadratic Graphs

A complete sketch of $f(x) = ax^2 + bx + c$ shows:

- The **shape** (upward or downward parabola).
- The **turning point** from completed square form.
- The **y -intercept**: substitute $x = 0$ to get $(0, c)$.
- The **x -intercepts** (if any): solve $ax^2 + bx + c = 0$.
- The **axis of symmetry** : $x = -b/(2a)$.

The diagram below shows all five features for $y = x^2 - 4x + 3$.

<QuadraticSketchDiagram />

The Discriminant and Nature of Roots

For $ax^2 + bx + c = 0$, the **discriminant** is $\Delta = b^2 - 4ac$.

Value of Δ	Nature of roots	Graph interpretation
$\Delta > 0$	Two distinct real roots	Parabola crosses x -axis twice
$\Delta = 0$	One repeated (equal) real root	Parabola touches x -axis once
$\Delta < 0$	No real roots	Parabola does not meet x -axis

Example

Determine the values of k for which $kx^2 + (k - 3)x + 1 = 0$ has no real roots.

For no real roots: $\Delta < 0$.

$$\Delta = (k - 3)^2 - 4k(1) = k^2 - 6k + 9 - 4k = k^2 - 10k + 9$$

Set $\Delta < 0$: $(k - 1)(k - 9) < 0$, so $1 < k < 9$.

Also, for the equation to be quadratic, $k \neq 0$. Since $k = 0$ is outside the interval $1 < k < 9$, the final answer is $1 < k < 9$.

Methods of Solving Quadratic Equations

Factorisation

is fastest when the quadratic factors cleanly over the integers. Look for two numbers whose product is

ac and sum is b .

Completing the square works for any quadratic and gives the exact roots in surd form.

Quadratic formula: For $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Remember

When a question says "solve," any valid method is acceptable. When it says "complete the square" or "express in the form

$a(x + h)^2 + k$," you must use that specific method.

Equations Reducible to a Quadratic

Some equations become quadratic after a substitution.

Common substitutions:

Equation type	Substitution	Resulting quadratic
[Math: $x^4 - 5x^2 + 4 = 0$]	[Math: $u = x^2$]	[Math: $u^2 - 5u + 4 = 0$]
[Math: $x - 5\sqrt{x} + 6 = 0$]	[Math: $u = \sqrt{x}$]	[Math: $u^2 - 5u + 6 = 0$]
[Math: $2^{2x} - 3(2^x) - 4 = 0$]	[Math: $u = 2^x$]	[Math: $u^2 - 3u - 4 = 0$]

Example

Solve $x - 5\sqrt{x} + 6 = 0$.

Let $u = \sqrt{x}$ (so $u \geq 0$ and $x = u^2$):

$$u^2 - 5u + 6 = 0 \Rightarrow (u - 2)(u - 3) = 0$$

so $u = 2$ or $u = 3$. Since $u = \sqrt{x}$: $x = 4$ or $x = 9$.

Check: $4 - 5(2) + 6 = 0$ and $9 - 5(3) + 6 = 0$

Remember

An **extraneous solution**

is a value produced by the algebra that does not satisfy the original equation. It arises because a substitution (like

$u = \sqrt{x}$, which requires $u \geq 0$

) imposes a domain restriction that the factored quadratic silently ignores. Always verify every back-substituted solution in the original equation. Any value that fails the check must be marked

(Invalid) and discarded — the algebra will not warn you automatically.

Relationships Between Roots and Coefficients

For $ax^2 + bx + c = 0$ with roots α and β :

$$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

These allow you to find expressions involving α and β without solving for them individually.

Expression	How to find it
$[\text{Math: } \alpha^2 + \beta^2]$	$[\text{Math: } (\alpha + \beta)^2 - 2\alpha\beta]$
$[\text{Math: } \alpha^2 \beta + \alpha\beta^2]$	$[\text{Math: } \alpha\beta(\alpha + \beta)]$
$[\text{Math: } \frac{1}{\alpha} + \frac{1}{\beta}]$	$[\text{Math: } \frac{\alpha + \beta}{\alpha\beta}]$
$[\text{Math: } (\alpha - \beta)^2]$	$[\text{Math: } (\alpha + \beta)^2 - 4\alpha\beta]$

Example

The roots of $3x^2 - 5x + 2 = 0$ are α and β . Find $\alpha^2 + \beta^2$.

$$\alpha + \beta = \frac{5}{3}, \quad \alpha\beta = \frac{2}{3}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{25}{9} - \frac{4}{3} = \frac{25}{9} - \frac{12}{9} = \frac{13}{9}$$

Forming a New Quadratic with Given Roots (Vieta's Root Theorem)

If a question asks for a quadratic whose roots are some transformation of α and β , find the new sum and product, then substitute into:

$$x^2 - (\text{Sum of Roots})x + (\text{Product of Roots}) = 0$$

If the coefficients are not integers, multiply through by the appropriate constant so that $a, b, c \in \mathbb{Z}$.

Example

The roots of $2x^2 - 3x + 1 = 0$ are α and β

. Find the quadratic with integer coefficients whose roots are α^2 and β^2 .

From the original:

$$\alpha + \beta = \frac{3}{2}$$

$$\alpha\beta = \frac{1}{2}$$

New sum:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{9}{4} - 1 = \frac{5}{4}$$

New product:

$$\alpha^2\beta^2 = (\alpha\beta)^2 = \frac{1}{4}$$

Substituting into the formula:

$$x^2 - \frac{5}{4}x + \frac{1}{4} = 0$$

Coefficients are not integers. Multiply through by 4:

$$4x^2 - 5x + 1 = 0$$

Simultaneous Equations: One Linear, One Non-Linear

The standard method is substitution: express one variable from the linear equation, then substitute into the quadratic (or circle) equation.

 **Example**

Solve the system: $y = x + 1$ and $x^2 + y^2 = 13$.

Substitute $y = x + 1$ into the circle:

$$x^2 + (x + 1)^2 = 13$$

$$x^2 + x^2 + 2x + 1 = 13$$

$$2x^2 + 2x - 12 = 0 \implies x^2 + x - 6 = 0 \implies (x + 3)(x - 2) = 0$$

$x = -3$ gives $y = -2$; $x = 2$ gives $y = 3$.

Solutions: $(-3, -2)$ and $(2, 3)$.

 **Exam Tip**

When a line intersects a circle, the discriminant of the resulting quadratic tells you about the intersection.

$\Delta > 0$: two points; $\Delta = 0$: the line is a tangent; $\Delta < 0$: no intersection.