Sequences and Series

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Sequences and Series

A **sequence** is an ordered list of numbers following a rule. A **series** is the sum of the terms of a sequence. Paper 01 allocates three items to this section; Paper 02 frequently integrates sequences into multi-part algebra questions.

Notation

The terms of a sequence are written $u1, u2, u3, \ldots, un$, where un is the general term (also called the nth term). The sum of the first n terms is written Sn.

Sigma notation compresses series:

$$S_n = \sum_{r=1}^{n} u_r = u_1 + u_2 + \cdots + u_n$$

The letter r is the index of summation. Key rules: $\sum(ur + vr) = \sum ur + \sum vr$ and $\sum c\,ur = c\sum ur$.

Arithmetic Sequences

An **arithmetic sequence**has a constant difference d between consecutive terms.

$$u_1, \quad u_1 + d, \quad u_1 + 2d, \quad \ldots$$

The nth term is:

$$u_n = a + (n-1)d$$

where $a = u_1$ is the first term and d is the common difference (which may be negative).

$u2 - u1 = u3 - u2 = d$

Identifying arithmetic sequences:check that (constant).

📋 **Example**

Find the 25th term of the sequence $7, 11, 15, 19, \ldots$

$a = 7$, $d = 4$. $u_{25} = 7 + 24(4) = 7 + 96 = 103$

📋 **Example**

An arithmetic sequence has $u5 = 17$ and $u12 = 38$. Find a and d.

$$u_{12} - u_5 = 7d = 38 - 17 = 21 \Rightarrow d = 3$$

$$a = u_5 - 4d = 17 - 12 = 5$$

Sum of an Arithmetic Series

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

An equivalent form when the last term $l = u_n$ is known:

$$S_n = \frac{n}{2}(a + l)$$

📋 **Example**

Find the sum of the first 20 terms of $5, 9, 13, 17, \ldots$

$a = 5$, $d = 4$, $n = 20$.

$$S_{20} = \frac{20}{2}[2(5) + 19(4)] = 10[10 + 76] = 10 \times 86 = 860$$

Divergence of Arithmetic Series

Every arithmetic series diverges unless $d = 0$. As n grows without bound, S_n grows without bound (positively or negatively). There is no finite sum to infinity for an arithmetic series.

Geometric Sequences

A **geometric sequence**has a constant ratio r between consecutive terms.

$$u_1, \quad u_1 r, \quad u_1 r^2, \quad \ldots$$

The n th term is:

$$u_n = ar^{n-1}$$

Identifying geometric sequences:check that $u2/u1 = u3/u2 = r$ (constant).

> 📋 **Example**
> Find the 8th term of $3, 6, 12, 24, \ldots$
>
> $a = 3$, $r = 2$. $\quad u_8 = 3 \times 2^7 = 3 \times 128 = 384$

Sum of a Geometric Series

For $r \neq 1$:

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad (\text{use when } r > 1)$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad (\text{use when } |r| < 1\text{, avoids negatives})$$

Both forms are equivalent. Choose whichever keeps the numerator positive to reduce sign errors.

> 📋 **Example**
> Find the sum of the first 6 terms of $2, 6, 18, 54, \ldots$
>
> $a = 2$, $r = 3$, $n = 6$.
>
> $$S_6 = \frac{2(3^6 - 1)}{3 - 1} = \frac{2(729 - 1)}{2} = 728$$

Study Vault

Convergence and Sum to Infinity

A geometric series **converges**(has a finite sum to infinity) if and only if $|r| < 1$.

When $|r| \geq 1$, the terms do not approach zero and the series **diverges**.

For a convergent geometric series:

$$S_\infty = \frac{a}{1-r} \qquad (|r| < 1)$$

📋 **Example**

Find the sum to infinity of $12 + 6 + 3 + 1.5 + \cdots$

$a = 12$, $r = \frac{1}{2}$. Since $|r| = \frac{1}{2} < 1$, the series converges.

$$S_\infty = \frac{12}{1 - \frac{1}{2}} = \frac{12}{\frac{1}{2}} = 24$$

📝 **Exam Tip**

Before applying S_∞, always state explicitly that $|r| < 1$
. Applying the formula to a divergent series ($|r| \geq 1$
) gives a meaningless result and loses method marks.

Finding the Common Ratio from a Convergence Condition

📋 **Example**

The first term of a geometric series is 8 and its sum to infinity is 20. Find the common ratio.

$$S_\infty = \frac{a}{1-r} \Rightarrow 20 = \frac{8}{1-r} \Rightarrow 1 - r = \frac{8}{20} = \frac{2}{5} \Rightarrow r = \frac{3}{5}$$

Check: $|r| = \frac{3}{5} < 1$

Comparing Arithmetic and Geometric

Feature	Arithmetic	Geometric
Pattern	Add constant [Math: d]	Multiply by constant [Math: r]
[Math: n]th term	[Math: a + (n-1)d]	[Math: ar^{n-1}]
Sum to [Math: n] terms	[Math: \frac{n}{2}[2a+(n-1)d]]	[Math: \frac{a(r^n-1)}{r-1}]
Converges?	No (unless [Math: d=0])	Yes, if [Math: \lvert r\rvert<1]
Sum to infinity	Does not exist	[Math: \frac{a}{1-r}] when [Math: \lvert r\rvert<1]

Real-World Applications

Repeated percentage change produces a geometric sequence. A quantity starting at A and changing by a fixed factor k each period has value Ak^{n-1} after $(n-1)$ periods.

📋 **Example**

A car is purchased for 20,000 and depreciates by 15% each year. Find its value after 4 years.

Each year the value is multiplied by $1-0.15=0.85$.

Value after 4 years: $20{,}000 \times (0.85)^4 \approx 10{,}440$

📋 **Example**

An investment of 5,000 earns compound interest at 6% per annum. How many complete years until it doubles?

After n years: value $= 5000 \times (1.06)^n$. Set this $\geq 10{,}000$:

$$(1.06)^n \geq 2$$

$$n \log 1.06 \geq \log 2 \Rightarrow n \geq \frac{\log 2}{\log 1.06} \approx 11.9$$

The investment first exceeds double after **12 complete years**.