

# Sequences and Series

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## Sequences and Series

A **sequence** is an ordered list of numbers following a rule. A **series** is the sum of the terms of a sequence. Paper 01 allocates three items to this section; Paper 02 frequently integrates sequences into multi-part algebra questions.

### Notation

The terms of a sequence are written  $u_1, u_2, u_3, \dots, u_n$ , where  $u_n$  is the general term (also called the  $n$ th term). The sum of the first  $n$  terms is written  $S_n$ .

Sigma notation compresses series:

$$S_n = \sum_{r=1}^n u_r = u_1 + u_2 + \dots + u_n$$

The letter  $r$  is the index of summation. Key rules:  $\sum(ur + vr) = \sum ur + \sum vr$  and  $\sum c ur = c \sum ur$ .

### Arithmetic Sequences

An **arithmetic sequence** has a constant difference  $d$  between consecutive terms.

$$u_1, \quad u_1 + d, \quad u_1 + 2d, \quad \dots$$


The  $n$ th term is:

$$u_n = a + (n - 1)d$$

where  $a = u_1$  is the first term and  $d$  is the common difference (which may be negative).


$$u_2 - u_1 = u_3 - u_2 = d$$

**Identifying arithmetic sequences:** check that (constant).

 **Example**

Find the 25th term of the sequence **7, 11, 15, 19, ...**

$$a = 7, d = 4. \quad u_{25} = 7 + 24(4) = 7 + 96 = 103$$

 **Example**

An arithmetic sequence has  **$u_5 = 17$**  and  **$u_{12} = 38$** . Find  **$a$**  and  **$d$** .

$$u_{12} - u_5 = 7d = 38 - 17 = 21 \Rightarrow d = 3$$


$$a = u_5 - 4d = 17 - 12 = 5$$

## Sum of an Arithmetic Series

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

An equivalent form when the last term  **$l = u_n$**  is known:

$$S_n = \frac{n}{2}(a + l)$$

 **Example**

Find the sum of the first 20 terms of **5, 9, 13, 17, ...**

$$a = 5, d = 4, n = 20.$$

$$S_{20} = \frac{20}{2}[2(5) + 19(4)] = 10[10 + 76] = 10 \times 86 = 860$$

## Divergence of Arithmetic Series

Every arithmetic series diverges unless  **$d = 0$** . As  **$n$**  grows without bound,  **$S_n$**  grows without bound (positively or negatively). There is no finite sum to infinity for an arithmetic series.

## Geometric Sequences

A **geometric sequence** has a constant ratio  $r$  between consecutive terms.

$$u_1, \quad u_1r, \quad u_1r^2, \quad \dots$$

The  $n$ th term is:

$$u_n = ar^{n-1}$$

**Identifying geometric sequences:** check that  $u_2/u_1 = u_3/u_2 = r$  (constant).

### Example

Find the 8th term of **3, 6, 12, 24, ...**

$$a = 3, \quad r = 2. \quad u_8 = 3 \times 2^7 = 3 \times 128 = 384$$

## Sum of a Geometric Series

For  $r \neq 1$ :

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad (\text{use when } r > 1)$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad (\text{use when } |r| < 1, \text{ avoids negatives})$$

Both forms are equivalent. Choose whichever keeps the numerator positive to reduce sign errors.

### Example

Find the sum of the first 6 terms of **2, 6, 18, 54, ...**

$$a = 2, \quad r = 3, \quad n = 6.$$

$$S_6 = \frac{2(3^6 - 1)}{3 - 1} = \frac{2(729 - 1)}{2} = 728$$

## Convergence and Sum to Infinity

A geometric series **converges** (has a finite sum to infinity) if and only if  $|r| < 1$ .

When  $|r| \geq 1$ , the terms do not approach zero and the series **diverges**.

For a convergent geometric series:

$$S_{\infty} = \frac{a}{1-r} \quad (|r| < 1)$$

### Example

Find the sum to infinity of  $12 + 6 + 3 + 1.5 + \dots$

$a = 12$ ,  $r = \frac{1}{2}$ . Since  $|r| = \frac{1}{2} < 1$ , the series converges.

$$S_{\infty} = \frac{12}{1 - \frac{1}{2}} = \frac{12}{\frac{1}{2}} = 24$$

### Exam Tip

Before applying  $S_{\infty}$ , always state explicitly that  $|r| < 1$ . Applying the formula to a divergent series ( $|r| \geq 1$ ) gives a meaningless result and loses method marks.

## Finding the Common Ratio from a Convergence Condition

### Example

The first term of a geometric series is 8 and its sum to infinity is 20. Find the common ratio.

$$S_{\infty} = \frac{a}{1-r} \Rightarrow 20 = \frac{8}{1-r} \Rightarrow 1-r = \frac{8}{20} = \frac{2}{5} \Rightarrow r = \frac{3}{5}$$

Check:  $|r| = \frac{3}{5} < 1$

## Comparing Arithmetic and Geometric

Feature	Arithmetic	Geometric
Pattern	Add constant [Math: d]	Multiply by constant [Math: r]
[Math: n]th term	[Math: a + (n-1)d]	[Math: ar^{n-1}]
Sum to [Math: n] terms	[Math: \frac{n}{2}[2a+(n-1)d]]	[Math: \frac{a(r^n-1)}{r-1}]
Converges?	No (unless [Math: d=0])	Yes, if [Math: \vert r \vert < 1]
Sum to infinity	Does not exist	[Math: \frac{a}{1-r}] when [Math: \vert r \vert < 1]

## Real-World Applications

Repeated percentage change produces a geometric sequence. A quantity starting at  $A$  and changing by a fixed factor  $k$  each period has value  $Ak^{n-1}$  after  $(n-1)$  periods.

### Example

A car is purchased for 20,000 and depreciates by 15% each year. Find its value after 4 years. Each year the value is multiplied by  $1 - 0.15 = 0.85$ .

Value after 4 years:  $20,000 \times (0.85)^4 \approx 10,440$

### Example

An investment of 5,000 earns compound interest at 6% per annum. How many complete years until it doubles?

After  $n$  years: value =  $5000 \times (1.06)^n$ . Set this  $\geq 10,000$ :

$$(1.06)^n \geq 2$$

$$n \log 1.06 \geq \log 2 \Rightarrow n \geq \frac{\log 2}{\log 1.06} \approx 11.9$$

The investment first exceeds double after **12 complete years**.