Surds, Indices, and Logarithms

Matthew Williams • Add Math • May 16, 2026

Surds, Indices, and Logarithms

These three ideas sit in the same syllabus section because they are deeply connected: fractional indices are roots, logarithms reverse exponentiation, and surds express exact irrational values. Together they appear in four Paper 01 items and regularly in Paper 02.

Surds

A **surd**

is an irrational root that cannot be simplified to a whole number or fraction. Additional Mathematics prefers

exact forms: writing $\sqrt{3}$ is always better than the decimal approximation $1.732\ldots$ unless a question specifically asks for a decimal answer.

The two core laws are:

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (b \neq 0)$$

Critical: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. This false identity is the most common surd error.

Simplifying Surds

Rewrite the radicand as (largest perfect square factor) $\times$ (remaining factor), then take the root of the perfect square.

> 📋 **Example**
> $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$
> $\sqrt{50x^5} = \sqrt{25x^4 \cdot 2x} = 5x^2\sqrt{2x}$

Adding and Subtracting Surds

Only **like surds** (same radicand) can be combined. Always simplify each surd first.

> 📋 **Example**
> $\sqrt{12}+\sqrt{27}=2\sqrt{3}+3\sqrt{3}=5\sqrt{3}$
> $\sqrt{8}+\sqrt{18}-\sqrt{2}=2\sqrt{2}+3\sqrt{2}-\sqrt{2}=4\sqrt{2}$

Multiplying Surds and Expanding Brackets

Multiply coefficients together and radicands together, then simplify.

> 📋 **Example**
> $(3+\sqrt{5})(2-\sqrt{5})=6-3\sqrt{5}+2\sqrt{5}-5=1-\sqrt{5}$

Rationalising the Denominator

Monomial denominator: multiply numerator and denominator by the surd.

$$\frac{5}{\sqrt{3}}=\frac{5}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{5\sqrt{3}}{3}$$

Binomial denominator: multiply by the **conjugate** (change the sign between the two terms). The denominator becomes a difference of squares with no surd.

> 📋 **Example**
> Rationalise $\frac{1+\sqrt{2}}{3-\sqrt{2}}$.
>
> Conjugate of $3-\sqrt{2}$ is $3+\sqrt{2}$.
>
> $$\frac{1+\sqrt{2}}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{(1+\sqrt{2})(3+\sqrt{2})}{9-2}$$
>
> Numerator: $3+\sqrt{2}+3\sqrt{2}+2=5+4\sqrt{2}$
>
> $$=\frac{5+4\sqrt{2}}{7}$$

📝 **Exam Tip**

A binomial denominator always signals: use the conjugate. The product $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$ eliminates the surd from the denominator completely.

Laws of Indices

The index (exponent) laws apply to any base $a > 0$:

Law	Statement
Multiplication	[Math: a^m \times a^n = a^{m+n}]
Division	[Math: a^m \div a^n = a^{m-n}]
Power of a power	[Math: (a^m)^n = a^{mn}]
Zero index	[Math: a^0 = 1]
Negative index	[Math: a^{-n} = \dfrac{1}{a^n}]
Fractional index	[Math: a^{1/n} = \sqrt[n]{a}]
General fractional	[Math: a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}]

Study Vault

📋 **Example**

$$81^{3/4} = \left(\sqrt[4]{81}\right)^3 = 3^3 = 27$$

$$\frac{a^{-2}b^3}{ab^{-1}} = a^{-2-1}b^{3-(-1)} = a^{-3}b^4 = \frac{b^4}{a^3}$$

🧠 **Remember**

The multiplication law $a^m \times a^n = a^{m+n}$ only applies when the **bases are the same**. You cannot combine $2^3 \times 3^2$ using index laws.

Solving Exponential Equations (Same Base)

If both sides can be written with the same base, equate the exponents.

> 📋 **Example**
> Solve $3^{2x-1} = 81$.
>
> $81 = 3^4$, so $3^{2x-1} = 3^4 \Rightarrow 2x - 1 = 4 \Rightarrow x = \frac{5}{2}$

Exponential Equations Reducible to Quadratic

Equations like $2^{2x} - 3(2^x) - 4 = 0$ become quadratic with the substitution $u = 2^x$ (since $2^{2x} = (2^x)^2 = u^2$).

> 📋 **Example**
> Solve $2^{2x+1} - 3(2^x) - 2 = 0$.
>
> Rewrite: $2 \cdot 2^{2x} - 3 \cdot 2^x - 2 = 0$.
>
> Let $u = 2^x$: $2u^2 - 3u - 2 = 0 \Rightarrow (2u+1)(u-2) = 0$.
>
> $u = -\frac{1}{2}$ or $u = 2$.
>
> **(Invalid):**Since $2^x > 0$ for all real x, $u = -\frac{1}{2}$ cannot be a value of 2^x for any real x. This is an extraneous solution introduced by the substitution — discard it.
>
> $2^x = 2 \Rightarrow x = 1$.

Logarithms

A **logarithm** answers the question: "What power must the base be raised to in order to produce this number?"

$$\log_a b = c \iff a^c = b \qquad (a > 0,\ a \neq 1,\ b > 0)$$

The logarithm $\log x$ (no base written) means $\log_{10} x$ at CSEC level.

Key values to know:

- $\log_a 1 = 0$ (since $a^0 = 1$)
- $\log_a a = 1$ (since $a^1 = a$)

Laws of Logarithms

Law	Statement
Product	[Math: \loga(xy) = \loga x + \log_a y]
Quotient	[Math: \loga\!\left(\dfrac{x}{y}\right) = \loga x - \log_a y]
Power	[Math: \loga(x^n) = n\loga x]

These laws mirror the index laws, which is why indices and logarithms are studied together.

📝 **Exam Tip**
The change-of-base formula is excluded from the CSEC syllabus. All required computations use the laws above, the definition $\log_a b = c \iff a^c = b$, and substitution of known values.

Simplifying Logarithmic Expressions

📋 **Example**
Simplify $2\log x - \log y + \log(x^2 y)$.

$$= \log x^2 - \log y + \log(x^2 y) = \log\left(\frac{x^2 \cdot x^2 y}{y}\right) = \log x^4$$

Solving Logarithmic Equations

Strategy 1: Combine logarithms into one using the laws, then convert to exponential form.

📋 **Example**

Solve $\log 3(x+1) + \log 3(x-2) = 2$.

$$\log_3[(x+1)(x-2)] = 2 \Rightarrow (x+1)(x-2) = 3^2 = 9$$

$$x^2 - x - 2 = 9 \Rightarrow x^2 - x - 11 = 0 \Rightarrow x = \frac{1 \pm \sqrt{45}}{2} = \frac{1 \pm 3\sqrt{5}}{2}$$

The two roots are $x = \frac{1+3\sqrt{5}}{2} \approx 3.85$ and $x = \frac{1-3\sqrt{5}}{2} \approx -2.85$.

(Invalid): $\log_3$ requires all arguments to be strictly positive. $x - 2 > 0$ demands $x > 2$, so $x \approx -2.85$ is an extraneous solution — discard it.

The only valid solution is $x = \frac{1+3\sqrt{5}}{2}$.

Strategy 2: If logarithms on both sides have the same base, equate arguments.

$$\log_a f(x) = \log_a g(x) \implies f(x) = g(x) \quad \text{(then verify both arguments are positive)}$$

Solving [Math: a^x = b] Using Logarithms

When bases cannot be matched, apply $\log$ to both sides and use the power law.

📋 **Example**

Solve $5^x = 37$.

$$\log(5^x) = \log 37 \Rightarrow x \log 5 = \log 37 \Rightarrow x = \frac{\log 37}{\log 5} \approx 2.24$$

🧠 **Remember**

Always check that all logarithm arguments are strictly positive in your solution. Extraneous roots are common in logarithmic equations because the algebra does not detect them automatically.

Linearisation Using Logarithms

Some real-world relationships follow an exponential or power law rather than a straight line. A linearisation question is usually signalled by a phrase like "a graph of $\log y$ against x gives a straight line" or "a graph of $\log y$ against $\log x$ gives a straight line." Applying logarithms to both sides of the original equation converts it into a linear form $Y = mX + c$, so the gradient and intercept of the graph can be read off to recover the original constants.

Original form	Take log of both sides	Linear form
[Math: y = ab^x]	[Math: \log y = \log a + x \log b]	[Math: Y = c + mX] where [Math: Y = \log y], [Math: X = x], [Math: m = \log b], [Math: c = \log a]
[Math: y = ax^n]	[Math: \log y = \log a + n\log x]	[Math: Y = c + mX] where [Math: Y = \log y], [Math: X = \log x], [Math: m = n], [Math: c = \log a]

📋 **Example**

Variables x and y are related by $y = ab^x$. A straight-line graph of $\log y$ against x has gradient 0.3 and y-intercept 1.2. Find a and b.

From the linear form: gradient $= \log b = 0.3$, so $b = 10^{0.3} \approx 2.00$.

y-intercept $= \log a = 1.2$, so $a = 10^{1.2} \approx 15.8$.

📝 **Exam Tip**

In linearisation questions, identify whether the graph plots $\log y$ vs x (suggesting $y = ab^x$) or $\log y$ vs $\log x$ (suggesting $y = ax^n$). This determines which form to use and what the gradient and intercept represent.