Vectors

Matthew Williams • Add Math • May 16, 2026

Vectors

A **vector** has both magnitude and direction. A **scalar** has magnitude only. Velocity, displacement, and force are vectors; speed, mass, and temperature are scalars.

Notation

Vectors are written in bold (**a**) in typeset text, or underlined ($\underline{a}$) in handwriting. In two dimensions, a vector is expressed as a **column vector**

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

or in terms of the standard unit vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$.

A **position vector** $\overrightarrow{OP}$ describes the location of point P relative to the origin O.

Operations on Vectors

Addition and subtraction: add or subtract corresponding components.

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

Scalar multiplication:multiply each component by the scalar k. This scales the magnitude by $|k|$ and reverses direction if $k < 0$.

$$k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

Equal vectors have the same magnitude and direction; position does not matter.

Magnitude

The magnitude (length) of $\mathbf{v} = \binom{x}{y}$ is:

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

This is Pythagoras applied to the horizontal and vertical components.

📋 **Example**

$\left|\begin{pmatrix} 5 \\ -12 \end{pmatrix}\right| = \sqrt{25 + 144} = \sqrt{169} = 13$

Direction of a Vector

The **direction**of a vector is the angle it makes with the positive x -axis, measured anticlockwise. For a vector with components (x, y) :

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$\arctan$ returns values between $-90°$ and $90°$, so the quadrant must be checked. For a vector in Quadrant II or III, add $180°$ to the result.

📋 **Example**

Find the direction of $\mathbf{v} = -3\mathbf{i} + 3\mathbf{j}$.

$\arctan\left(\frac{3}{-3}\right) = \arctan(-1) = -45°$.

Since $x < 0$ and $y > 0$ the vector is in Quadrant II, so direction $= -45° + 180° = 135°$.

Unit Vectors

A **unit vector**has magnitude 1. To find the unit vector in the direction of $\mathbf{a}$:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

📋 **Example**

Find the unit vector in the direction of $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

$|\mathbf{a}| = \sqrt{9 + 16} = 5$

$\hat{\mathbf{a}} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$

Displacement Vectors

The vector from point A to point B is:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

Equivalently, $\overrightarrow{AB} = B - A$ (coordinates of B minus coordinates of A). The order matters: $\overrightarrow{AB} = -\overrightarrow{BA}$.

📋 **Example**

If $A = (2, -1)$ and $B = (7, 4)$:

$$\overrightarrow{AB} = \begin{pmatrix} 7 - 2 \\ 4 - (-1) \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Parallel and Collinear Vectors

Two vectors are **parallel** if one is a scalar multiple of the other: $\mathbf{b} = k\mathbf{a}$ for some scalar k.

Three points A, B, C are **collinear** if $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$ (and both share point A).

📋 **Example**

Show that $A = (1, 2)$, $B = (3, 6)$, $C = (5, 10)$ are collinear.

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Since $\overrightarrow{AC} = 2\,\overrightarrow{AB}$ and they share point A, the three points are collinear.

The Scalar (Dot) Product

The scalar product of $\mathbf{a} = \binom{a1}{a2}$ and $\mathbf{b} = \binom{b1}{b2}$ is defined two ways:

Component form:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

Geometric form:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between the vectors ($0 \le \theta \le \pi$).

The result is a **scalar** (a number), not a vector.

Finding the Angle Between Two Vectors

Combining both definitions:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

📋 **Example**

Find the angle between $\mathbf{a} = \binom{1}{2}$ and $\mathbf{b} = \binom{3}{-1}$.

$\mathbf{a} \cdot \mathbf{b} = (1)(3) + (2)(-1) = 1$

$|\mathbf{a}| = \sqrt{5}, \ |\mathbf{b}| = \sqrt{10}$

$$\cos\theta = \frac{1}{\sqrt{5} \cdot \sqrt{10}} = \frac{1}{\sqrt{50}} = \frac{1}{5\sqrt{2}} \Rightarrow \theta = \arccos\left(\frac{1}{5\sqrt{2}}\right) \approx 81.9^\circ$$

Perpendicular Vectors

Two non-zero vectors are **perpendicular** if and only if their dot product is zero:

$$\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a} \perp \mathbf{b}$$

This follows directly from the geometric form: $\cos 90^\circ = 0$.

📋 **Example**

Are $\mathbf{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ perpendicular?

$\mathbf{p} \cdot \mathbf{q} = (4)(3) + (-3)(4) = 12 - 12 = 0$. Yes, they are perpendicular.

📝 **Exam Tip**

Paper 02 often asks you to find an unknown component given that two vectors are perpendicular. Set the dot product equal to zero and solve for the unknown.

Properties of the Scalar Product

- Commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ (useful for finding $|\mathbf{a}|$ without the square root formula when the vector is expressed algebraically)
- Distributive: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (used when expanding expressions involving sums of vectors)

Scalars vs Vectors: Common Examples

Scalar	Vector
Distance	Displacement
Speed	Velocity
Mass	Force
Temperature	Acceleration