Variation, Word Problems & Algebraic Identities

Amari Cross & Matthew Williams • Math • May 6, 2026

This page brings together the parts of algebra that usually require more than one step of thinking. Instead of only following a memorised procedure, you must decide how to translate a situation, combine equations, or justify that two expressions are always equal. That is why these topics are common in CSEC reasoning questions. For Paper 02, do not jump straight to the answer: define the variables, write the equation or relationship, solve carefully, and then interpret the result in words. The explanation is part of the mathematics.

What Are Non-Linear Equations?

Non-linear means at least one equation has a variable to a power higher than 1 (like x^2).

When you solve non-linear simultaneous equations, you're finding where a line intersects a curve (like a parabola).

Strategy: Substitution (Exactly Like Linear)

The method is the SAME as linear simultaneous equations:

- 1. Rearrange one equation to get a variable by itself
- 2. Substitute into the other equation
- 3. Solve (may be quadratic now)
- 4. Find the other variable

Example 1: Line + Parabola

📋 **Example**

Solve:

$$y = x + 3 \quad \text{...(1)}$$

$$y = x^2 - 2x + 1 \quad \text{...(2)}$$

Step 1: Equation (1) already has y isolated: $y = x + 3$

Step 2: Substitute into equation (2)

$$x + 3 = x^2 - 2x + 1$$

Step 3: Rearrange into standard form

$$0 = x^2 - 2x + 1 - x - 3$$

$$0 = x^2 - 3x - 2$$

Step 4: Check the discriminant

$$\Delta = b^2 - 4ac = (-3)^2 - 4(1)(-2) = 9 + 8 = 17$$

Since $\Delta > 0$, there are two distinct real solutions.

Step 5: Use the quadratic formula

$$x = \frac{-(-3) \pm \sqrt{17}}{2(1)} = \frac{3 \pm \sqrt{17}}{2}$$

$$x_1 = \frac{3 + \sqrt{17}}{2} \approx 3.56$$

$$x_2 = \frac{3 - \sqrt{17}}{2} \approx -0.56$$

Step 6: Find corresponding y values using equation (1)

- When $x \approx 3.56$: $y = 3.56 + 3 = 6.56$
- When $x \approx -0.56$: $y = -0.56 + 3 = 2.44$

Solutions: $(3.56, 6.56)$ and $(-0.56, 2.44)$ approximately

Or exactly: $\left(\frac{3 + \sqrt{17}}{2}, \frac{9 + \sqrt{17}}{2}\right)$ and $\left(\frac{3 - \sqrt{17}}{2}, \frac{9 - \sqrt{17}}{2}\right)$

<JSXGraph id="algebra-4" title="y = 2x + 1 (line) and y = x² - 2 (parabola)" boundingbox={[-4, 8, 4, -4]} height={360} code="board.create('functiongraph',[function(x){return 2x+1;},-4,4],{strokeColor:'#2563eb',strokeWidth:2,name:'y=2x+1',withLabel:true});board.create('functiongraph xx-2;},-4,4],{strokeColor:'#dc2626',strokeWidth:2,name:'y=x²-2',withLabel:true});" />

Example 2: Two Parabolas (Quadratic + Quadratic)

📋 **Example**

Solve:

[MathBlock]
y = x^2 + 2 \quad \text{...(1)}
[/MathBlock]

[MathBlock]
y = -x^2 + 4x - 1 \quad \text{...(2)}
[/MathBlock]

Step 1: Both have [Math: y] isolated

Step 2: Set them equal

[MathBlock]
x^2 + 2 = -x^2 + 4x - 1
[/MathBlock]

Step 3: Rearrange

[MathBlock]
x^2 + x^2 - 4x + 2 + 1 = 0
[/MathBlock]

[MathBlock]
2x^2 - 4x + 3 = 0
[/MathBlock]

Step 4: Check discriminant

[MathBlock]
\Delta = (-4)^2 - 4(2)(3) = 16 - 24 = -8
[/MathBlock]

Since [Math: \Delta < 0], there are **NO real solutions**.

This means the two parabolas don't intersect.

How Many Solutions?

The discriminant tells you:

- [Math: \Delta > 0]: Two intersection points (line crosses curve twice)
- [Math: \Delta = 0]: One intersection point (line is tangent to curve, just touches)
- [Math: \Delta < 0]: No real intersection points (curves don't meet)

🧠 **Remember**

Common mistake: After substituting and getting a quadratic, students forget to solve it completely.

Remember: you need BOTH [Math: x] and [Math: y] values for each solution!

- 1. Solve the quadratic to find all [Math: x] values
- 2. For EACH [Math: x] value, find the corresponding [Math: y] value
- 3. Write both coordinates: [Math: (x, y)]

Part 14: Word Problems

Setting Up Equations from Words

Read carefully and define variables!

📋 **Example**

"The sum of two numbers is 15. Their difference is 3. Find the numbers."

Let [Math: x] = first number, [Math: y] = second number

[MathBlock]
x + y = 15 \quad \text{...(1)}
[/MathBlock]

[MathBlock]
x - y = 3 \quad \text{...(2)}
[/MathBlock]

Add equations: [Math: 2x = 18 \Rightarrow x = 9]

From (1): [Math: y = 6]

Numbers are 9 and 6.

📋 **Example**

"A rectangle's length is 3 cm more than its width. Its area is 40 cm². Find dimensions."

Let [Math: w] = width, [Math: l = w + 3] = length

[MathBlock]
w(w + 3) = 40
[/MathBlock]

[MathBlock]
w^2 + 3w - 40 = 0
[/MathBlock]

[MathBlock]
(w + 8)(w - 5) = 0
[/MathBlock]

[Math: w = 5] (take positive), [Math: l = 8]

Dimensions: 5 cm × 8 cm

Part 15: Direct and Inverse Variation

Direct Variation — Things That Grow Together

Direct variation happens when one quantity increases and another quantity **also increases** at the same rate.

Real-world examples:

- More workers = More work gets done
- Longer you drive = Further distance you travel
- More pizzas you buy = Higher your bill
- More study hours = Higher your test score (usually!)

Key idea: If you double one thing, the other thing doubles too. If you triple one thing, the other thing triples too.

The Pattern

When [Math: y] varies directly as [Math: x]:

y = kx

Where:

- [Math: y] and [Math: x] are the two related quantities
- [Math: k] = **constant of proportionality** (the "multiplier")
- [Math: k] is always the **same ratio**: [Math: k = \frac{y}{x}]

In other words: The ratio [Math: \frac{y}{x}] never changes.

How to Solve Direct Variation Problems

Step 1: Write the formula: [Math: y = kx]

Step 2: Find [Math: k] using the given information

Step 3: Write the specific equation with that [Math: k]

Step 4: Use it to answer questions

📋 **Example**

"It costs 4 dollars per item. If you buy 3 items, it costs 12 dollars. How much for 5 items?"

Step 1: Cost varies directly with number of items

[MathBlock]
\text{Cost} = k \times \text{Items}
[/MathBlock]

Step 2: Find [Math: k] using known values:

[MathBlock]
12 = k \times 3
[/MathBlock]

[MathBlock]
k = 4 \text{ dollars per item}
[/MathBlock]

This makes sense! Each item costs 4 dollars.

Step 3: The equation is:

[MathBlock]
\text{Cost} = 4 \times \text{Items}
[/MathBlock]

Step 4: For 5 items:

[MathBlock]
\text{Cost} = 4 \times 5 = 20 \text{ dollars}
[/MathBlock]

Notice: When items went from 3 to 5 (multiply by 5/3), cost went from 12 to 20 (also multiply by 5/3). They move together!

Graph of Direct Variation

A direct variation graph is always a **straight line through the origin**.

- **Why through the origin?** Because when [Math: x = 0], then [Math: y = k(0) = 0]. Always!
- **Steeper line = larger [Math: k] (stronger relationship)**
- **Flatter line = smaller [Math: k] (weaker relationship)**

<JSXGraph id="algebra-direct" title="Direct variation: y = kx through origin" boundingbox={[-1, 8, 8, -1]} height={320} code="board.create('functiongraph',[function(x){return 2x;},-1,8],{strokeColor:'#2563eb',strokeWidth:2,name:'y=2x',withLabel:true});board.create('functiongraph',[f
0.5x;},-1,8],{strokeColor:'#16a34a',strokeWidth:2,name:'y=0.5x',withLabel:true});" />

Inverse Variation — Things That Work Against Each Other

Inverse variation happens when one quantity increases and another quantity **decreases**. They work in opposite directions.

Real-world examples:

- More workers on a job = Less time the job takes
- Faster you drive = Less time to reach destination
- More money you have = Fewer days you need to work
- Larger the pizza = Fewer pizzas you need to feed people
- Higher the price = Fewer people will buy it

Key idea: If you double one thing, the other thing gets cut in half. If you triple one thing, the other thing gets divided by 3.

The Pattern

When [Math: y] varies inversely as [Math: x]:

```
y = \frac{k}{x}
```

Where:

- [Math: y] and [Math: x] are the two related quantities
- [Math: k] = **constant of proportionality**
- The **product** [Math: x \times y] is always the same: [Math: xy = k]

In other words: When you multiply [Math: x] and [Math: y] together, you always get [Math: k].

How to Solve Inverse Variation Problems

Step 1: Write the formula: [Math: y = \frac{k}{x}] (or [Math: xy = k])

Step 2: Find [Math: k] using the given information

Step 3: Write the specific equation with that [Math: k]

Step 4: Use it to answer questions

📋 **Example**

"A job takes 12 hours with 2 workers. How long with 4 workers?"

Step 1: Time varies inversely with number of workers

$$\text{Time} = \frac{k}{\text{Workers}}$$

Step 2: Find k using known values:

$$12 = \frac{k}{2}$$

$$k = 12 \times 2 = 24 \text{ worker-hours}$$

This makes sense! The total amount of work is 24 worker-hours. No matter how many workers, the work stays the same.

Step 3: The equation is:

$$\text{Time} = \frac{24}{\text{Workers}}$$

Step 4: For 4 workers:

$$\text{Time} = \frac{24}{4} = 6 \text{ hours}$$

Notice: When workers doubled (2 ’4), time halved (12 ’6). They move opposite ways!

Check the product:

- With 2 workers: $2 \times 12 = 24$
- With 4 workers: $4 \times 6 = 24$

The product is always 24!

Graph of Inverse Variation

An inverse variation graph is a **hyperbola** (curved shape, like a stretched smile).

- **Never touches the axes:** The graph never crosses $x = 0$ or $y = 0$
- **Closer to one axis, farther from other:** When x is huge, y is tiny (and vice versa)
- **Two branches:** One in upper-right quadrant, one in lower-left quadrant (for positive k)

- **Curves smooth, never straight:** The relationship changes as you move along the curve

<JSXGraph id="algebra-inverse" title="Inverse variation: y = 4/x" boundingbox={[-1, 8, 8, -1]} height={320} code="board.create('functiongraph',[function(x){return 4/x;},0.5,8],{strokeColor:'#2563eb',strokeWidth:2,name:'y=4/x',withLabel:true});" />

Comparing Direct vs. Inverse

Aspect	Direct ([Math: y = kx])	Inverse ([Math: y = k/x])
Relationship	Both increase together	One increases, other decreases
Graph shape	Straight line through origin	Hyperbola (curved)
Constant	Ratio: [Math: y/x = k]	Product: [Math: xy = k]
Example	Pay = hourly rate × hours	Time = work ÷ workers
When one doubles	Other doubles	Other halves

📝 **Exam Tip**

Recognizing variation in exams:

Look for these phrases:

- "directly proportional" = direct variation: [Math: y = kx]
- "varies directly" = direct variation: [Math: y = kx]
- "inversely proportional" = inverse variation: [Math: y = k/x]
- "varies inversely" = inverse variation: [Math: y = k/x]

Strategy:

- 1. Identify which type (direct or inverse)
- 2. Write appropriate formula
- 3. Find [Math: k] from given information
- 4. Answer the question using that specific equation

Part 16: Proving Algebraic Identities — Showing Something Is Always True

What's an Identity?

An **identity** is an equation that is TRUE for ALL possible values of the variable(s).

Compare:

- **Equation:** [Math: $2x + 3 = 7$] (true only when [Math: $x = 2$])
- **Identity:** [Math: $2(x + 1) = 2x + 2$] (true for ANY value of [Math: x])

How to Prove an Identity

Goal: Show that the left side and right side are exactly the same thing.

Method: Transform one side (usually the more complicated side) into the other side using algebraic rules.

Key principle: You can only manipulate ONE side at a time. You cannot add/subtract the same thing from both sides like in equations.

Strategy

- 1. Look at both sides — which one looks more complicated?
- 2. Start with the complicated side
- 3. Use algebraic operations to simplify it
- 4. Try to make it look like the other side
- 5. When you've transformed one side into the other, you're done!

Example 1: Expanding to Prove

📋 **Example**

Prove: [Math: (a + b)^2 = a^2 + 2ab + b^2]

Which side is more complex? The left side has a squared binomial (more complex).

Start with left side and expand:

Step 1: Write the square as multiplication

[MathBlock]
(a+b)^2 = (a+b)(a+b)
[/MathBlock]

Step 2: Use FOIL

[MathBlock]
= a(a) + a(b) + b(a) + b(b)
[/MathBlock]

[MathBlock]
= a^2 + ab + ab + b^2
[/MathBlock]

Step 3: Combine like terms

[MathBlock]
= a^2 + 2ab + b^2
[/MathBlock]

Step 4: This is exactly the right side!

Left side = Right side Identity proven

What this proves: No matter what [Math: a] and [Math: b] are, [Math: (a+b)^2] will always equal [Math: a^2 + 2ab + b^2].

Example 2: Difference of Squares Pattern

📋 **Example**

Prove: [Math: (x-y)(x+y) = x^2 - y^2]

Which side is more complex? The left side has two binomials (more complex).

Start with left side and expand:

Step 1: Use FOIL on [Math: (x-y)(x+y)]

[MathBlock]
= x(x) + x(y) + (-y)(x) + (-y)(y)
[/MathBlock]

[MathBlock]
= x^2 + xy - xy - y^2
[/MathBlock]

Step 2: Combine like terms (notice [Math: xy - xy = 0])

[MathBlock]
= x^2 - y^2
[/MathBlock]

Step 3: This is exactly the right side!

Left side = Right side Identity proven

What this proves: Whenever you multiply a sum by a difference, the middle terms always cancel!

Example 3: Simplification to Prove

📋 **Example**

Prove: [Math: \frac{x^2 - 1}{x - 1} = x + 1] (for [Math: x \neq 1])

Which side is more complex? The left side has a fraction (more complex).

Start with left side and simplify:

Step 1: Factor the numerator (difference of squares)

[MathBlock]
\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1}
[/MathBlock]

Step 2: Cancel the common factor [Math: (x-1)]

[MathBlock]
= \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1
[/MathBlock]

Step 3: This is exactly the right side!

Left side = Right side Identity proven

Important: We require [Math: x \neq 1] because we can't divide by zero.

Example 4: Factoring to Prove

📋 **Example**

Prove: [Math: ab + 3a + 2b + 6 = (a + 2)(b + 3)]

Which side is more complex? The left side has four separate terms (more complex).

Start with left side and factor:

Step 1: Group the terms

[MathBlock]
ab + 3a + 2b + 6 = (ab + 3a) + (2b + 6)
[/MathBlock]

Step 2: Factor out common factors from each group

[MathBlock]
= a(b + 3) + 2(b + 3)
[/MathBlock]

Step 3: Factor out the common binomial

[MathBlock]
= (a + 2)(b + 3)
[/MathBlock]

Step 4: This is exactly the right side!

Left side = Right side Identity proven

What NOT to Do

🧠 **Remember**

Wrong approach: Don't start with the equality and try to manipulate both sides.

❌ Wrong:

[MathBlock]
x^2 + 2x + 1 = (x+1)^2
[/MathBlock]

[MathBlock]
\text{(subtract } x^2 \text{)} \quad 2x + 1 = (x+1)^2 - x^2
[/MathBlock]

Right:

[MathBlock]
\text{Start with } (x+1)^2 \text{ and show it simplifies to } x^2 + 2x + 1
[/MathBlock]

[MathBlock]
(x+1)^2 = (x+1)(x+1) = x^2 + x + x + 1 = x^2 + 2x + 1
[/MathBlock]

Proof Strategy Summary

- 1. **Identify the complex side** (usually the one with brackets, fractions, or more terms)
- 2. **Transform that side** using:
- Expanding (FOIL, distributive law)
- Factoring
- Canceling common factors
- Combining like terms
- 3. **Simplify step-by-step** until it matches the other side
- 4. **Conclude:** "Left side = Right side "