Quadratic Equations & Graphs

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Quadratics describe situations where the variable is squared, so their graphs curve instead of forming straight lines. In CSEC Mathematics, quadratics connect algebra and graphs: you may be asked to solve an equation, identify roots, find a turning point, or interpret the shape of a parabola.

The important idea is that each method reveals something different. Factorising shows where the graph crosses the x-axis, completing the square shows the turning point, and the quadratic formula works even when simple factorising does not. When a question carries reasoning marks, explain why the method you chose fits the form of the quadratic.

Solving Quadratics by Factorization

Factorisation is the quickest method when the quadratic breaks neatly into brackets. Once the product equals zero, at least one bracket must be zero. This is called the zero product property, and it is the reason the bracketed factors become separate linear equations.

If $ax^2 + bx + c = 0$ factors to $(px + q)(rx + s) = 0$, then:

$$px + q = 0 \text{ or } rx + s = 0$$

📋 **Example**

$x^2 + 5x + 6 = 0$

Factor: $(x + 2)(x + 3) = 0$

$$x + 2 = 0 \Rightarrow x = -2$$

$$x + 3 = 0 \Rightarrow x = -3$$

Solutions: $x = -2$ or $x = -3$

Check: $(-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$

Solving by Completing the Square

Completing the square is slower than factorising, but it reveals the turning point directly. This is especially useful for graph questions and for quadratics that do not factor nicely.

Express $ax^2 + bx + c$ as a perfect square plus a constant. This method also helps find the **vertex**(minimum or maximum point).

General Method:

For $ax^2 + bx + c = 0$:

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

This gives **vertex form**: $a(x-h)^2 + k = 0$

Where:

- $h = -\frac{b}{2a}$ (x-coordinate of vertex)
- $k = c - \frac{b^2}{4a}$ (y-coordinate of vertex)
- Vertex (h, k) is the **minimum** (if $a > 0$) or **maximum** (if $a < 0$)

Example 1: Simple Case ([Math: a = 1])

In this first case, the coefficient of x^2

is 1, so you can complete the square directly. The number added to the left must also be added to the right because the equation must stay balanced.

$x^2 + 6x - 7 = 0$

Step 1: Move constant to right side

$$x^2 + 6x = 7$$

Step 2: Take half the x-coefficient

$$\frac{b}{2} = \frac{6}{2} = 3$$

Step 3: Square it

$$\left(\frac{6}{2}\right)^2 = 9$$

Step 4: Add to both sides

$$x^2 + 6x + 9 = 7 + 9$$

Step 5: Factor left as perfect square

$$(x+3)^2 = 16$$

Step 6: Solve

$$x + 3 = \pm 4$$

$$x = -3 + 4 = 1 \text{ or } x = -3 - 4 = -7$$

Vertex (from form): $(x+3)^2 = 16$ means vertex at $x = -3$, and when $x = -3$: $(0)^2 = 16$ gives $y = 16$... wait, that's not right at 0. At $x = -3$: $y = (-3)^2 + 6(-3) - 7 = 9 - 18 - 7 = -16$

So vertex is at $(-3, -16)$ • **minimum point**

Check: $1^2 + 6(1) - 7 = 1 + 6 - 7 = 0$

Example 2: General Case ([Math: a `1])

When a is not 1, factor it out of the x^2 and x
terms first. This prevents you from adding the wrong amount to the expression.

$2x^2 + 8x - 10 = 0$

Step 1: Factor out leading coefficient from first two terms

$$2\left(x^2 + 4x\right) - 10 = 0$$

Step 2: Complete the square inside the brackets

- Half the x-coefficient: $\frac{4}{2} = 2$
- Square it: $2^2 = 4$

$$2\left(x^2 + 4x + 4\right) - 10 - 2(4) = 0$$

Note: We subtract $2 \times 4 = 8$ because we added 2×4 inside

Step 3: Factor as perfect square

$$2(x+2)^2 - 18 = 0$$

Step 4: Rearrange to find vertex

$$2(x+2)^2 = 18$$

$$(x+2)^2 = 9$$

Step 5: Solve

$$x + 2 = \pm 3$$

$$x = -2 + 3 = 1 \text{ or } x = -2 - 3 = -5$$

Vertex form: $2(x+2)^2 - 18 = 0$ can be written as $2(x-(-2))^2 + (-18) = 0$

So vertex at $(-2, -18)$ • **minimum point**(since $a = 2 > 0$)

Check: $2(1)^2 + 8(1) - 10 = 2 + 8 - 10 = 0$

Finding Min/Max Point Directly

Once in vertex form $a(x-h)^2 + k$, the **vertex is immediately** (h, k):

> 📋 **Example**
> **Find the vertex of** $y = 3(x-5)^2 + 7$
>
> Vertex = $(5, 7)$ • **maximum point**(since $a = 3 > 0$
> , wait no, that means upward so minimum)
>
> Actually: $a = 3 > 0$ means U-shape (opens upward), so $(5, 7)$ is the **minimum**
>
> **Find the vertex of** $y = -2(x+3)^2 - 4$
>
> Rewrite as $y = -2(x-(-3))^2 + (-4)$
>
> Vertex = $(-3, -4)$ • **maximum point**(since $a = -2 < 0$, opens downward)

Quadratic Formula

The quadratic formula is the most dependable solving method because it works for every quadratic in the form
$ax^2 + bx + c = 0$. It is useful when factorising is difficult or impossible.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For equation $ax^2 + bx + c = 0$

📋 **Example**

$2x^2 + 5x - 3 = 0$

$a = 2, b = 5, c = -3$

$$x = \frac{-5 \pm \sqrt{25 - 4(2)(-3)}}{2(2)}$$

$$= \frac{-5 \pm \sqrt{25 + 24}}{4}$$

$$= \frac{-5 \pm \sqrt{49}}{4}$$

$$= \frac{-5 \pm 7}{4}$$

$$x = \frac{-5 + 7}{4} = \frac{2}{4} = 0.5$$

$$x = \frac{-5 - 7}{4} = \frac{-12}{4} = -3$$

Solutions: $x = 0.5$ or $x = -3$

Check: $2(0.5)^2 + 5(0.5) - 3 = 0.5 + 2.5 - 3 = 0$

Discriminant

The discriminant is the part under the square root in the quadratic formula. Before solving fully, it tells you how many real answers to expect, which helps you check whether your final result makes sense.

The **discriminant** $\Delta = b^2 - 4ac$ tells us about the solutions:

- $\Delta > 0$: Two distinct real solutions
- $\Delta = 0$: One repeated real solution
- $\Delta < 0$: No real solutions (complex solutions)

📋 **Example**

$x^2 + 4x + 4 = 0$

$\Delta = 16 - 16 = 0$

One solution: $x = \frac{-4}{2} = -2$ (double root)

Actually: $(x+2)^2 = 0 \Rightarrow x = -2$ (repeated)

Quadratic Graphs

Parabolas

A parabola is the visual form of a quadratic. The roots, intercepts, vertex, and axis of symmetry are all different ways of describing the same curve.

A quadratic function $y = ax^2 + bx + c$ graphs as a parabola.

Properties:

- If $a > 0$: opens upward (U-shape), has **minimum** point
- If $a < 0$: opens downward (∩ shape), has **maximum** point
- Vertex = turning point (minimum or maximum)

📋 **Example**

$y = x^2 - 4x + 3$

Find vertex:

$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = 2$

$y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$

Vertex: $(2, -1)$ (minimum, since $a = 1 > 0$)

Find y-intercept: Set $x = 0 \Rightarrow y = 3$

Find x-intercepts: Set $y = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1, 3$

Vertex Form

Vertex form is designed for graph interpretation. It tells you the turning point immediately, so it is often more useful than standard form when the question asks for a maximum, minimum, or axis of symmetry.

$$y = a(x-h)^2 + k$$

Where (h, k) is the vertex.

Converting Standard Form to Vertex Form

Use completing the square:

$$y = ax^2 + bx + c$$

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Vertex: $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$

📋 **Example**

Convert $y = x^2 - 4x + 3$ to vertex form and find the vertex:

$a = 1, b = -4, c = 3$

$$y = 1\left(x^2 - 4x\right) + 3$$

$$y = 1\left(x^2 - 4x + 4\right) + 3 - 4$$

$$y = (x-2)^2 - 1$$

Vertex: $(2, -1)$ • **minimum point**(since $a = 1 > 0$)

Notice: $x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$

📋 **Example**

Convert $y = 2x^2 + 12x + 5$ to vertex form:

$a = 2, b = 12, c = 5$

$$y = 2\left(x^2 + 6x\right) + 5$$

Complete the square inside: half of 6 is 3, square is 9

$$y = 2\left(x^2 + 6x + 9\right) + 5 - 2(9)$$

$$y = 2(x+3)^2 + 5 - 18$$

$$y = 2(x+3)^2 - 13$$

Vertex: $(-3, -13)$ • **minimum point**(since $a = 2 > 0$)

This means:

- Parabola opens upward (U-shape)
- Lowest point is at $(-3, -13)$
- Minimum value of y is -13
- Axis of symmetry is $x = -3$

Quick Vertex Formula

The quick formula is a shortcut for the x

-coordinate of the turning point. It does not replace understanding the graph; after finding x

, you still substitute into the original equation to find the matching y value.

If you don't want to complete the square every time:

$$x_{\text{vertex}} = -\frac{b}{2a}$$

Then substitute back to find y_{vertex}:

$$y_{\text{vertex}} = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$$

📋 **Example**

Find the vertex of $y = 3x^2 - 12x + 7$:

$$x = -\frac{-12}{2(3)} = \frac{12}{6} = 2$$

$$y = 3(2)^2 - 12(2) + 7 = 12 - 24 + 7 = -5$$

Vertex: $(2, -5)$ **• minimum point**

📋 **Example**

$y = x^2 - 4x + 3$

Vertex: $(2, -1)$ (directly from the form!)

Since $a = 1 > 0$, parabola opens upward (minimum at $(2, -1)$)

Find y-intercept: Set $x = 0 \Rightarrow y = 3$

Find x-intercepts: Set $y = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1, 3$

Compare to standard form: $y = x^2 - 4x + 3$, same parabola!