Linear Functions & Graphs

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Linear functions connect algebra to visual patterns. The equation tells you how y changes as x changes, and the graph shows that relationship as a straight line.

In CSEC, graph questions may ask for gradient, intercepts, equations of lines, parallel or perpendicular lines, or graphical solutions to simultaneous equations. Do not only plot points; explain what the gradient and intercept mean. That helps with comprehension and reasoning marks.

What Is a Linear Function?

A **linear function** is a function where the graph is a straight line.

General form:

$$f(x) = mx + c$$

Or:

$$y = mx + c$$

Where:

- m = **slope** (gradient), how steep the line is
- c = **y-intercept**, where the line crosses the y-axis
- x = input (domain variable)
- y or $f(x)$ = output (range variable)

Understanding Slope (Gradient)

Slope measures how much y changes when x increases by 1.

$$m = \text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Interpretation:

$m > 0$

- : Line goes UP from left to right (positive slope)
- $m < 0$: Line goes DOWN from left to right (negative slope)
- $m = 0$: Horizontal line (flat, no change)
- m undefined: Vertical line

Examples of Different Slopes

Forms of Linear Equations

Form 1: Slope-Intercept Form (Most Useful)

$$y = mx + c$$

- Easy to identify: slope is m , y-intercept is c
- Easy to graph: plot $(0, c)$, then use slope to find more points

📋 **Example**

Graph $y = 2x - 3$**:**

- **Slope:** $m = 2$ (go up 2 for every right 1)
- **Y-intercept:** $c = -3$ (crosses y-axis at $(0, -3)$)

Plot key points:

- Start at $(0, -3)$
- Slope 2 means: right 1, up 2 'next point $(1, -1)$
- Continue: $(2, 1)$, $(3, 3)$, etc.

Form 2: Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Use this when you know:

- The slope m
- One point $(x1, y1)$ on the line

📋 **Example**

Find the equation of a line with slope 3 passing through $(2, 5)$:

Step 1: Use point-slope form

$$y - 5 = 3(x - 2)$$

Step 2: Expand

$$y - 5 = 3x - 6$$

Step 3: Rearrange to slope-intercept form

$$y = 3x - 6 + 5$$

$$y = 3x - 1$$

Form 3: Two-Point Form

When you know two points $(x1, y1)$ and $(x2, y2)$:

Step 1: Find slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 2: Use point-slope form with either point

📋 **Example**

Find the equation of a line through $(1, 3)$ and $(4, 12)$:

Step 1: Find slope

$$m = \frac{12 - 3}{4 - 1} = \frac{9}{3} = 3$$

Step 2: Use point-slope with first point $(1, 3)$

$$y - 3 = 3(x - 1)$$

$$y = 3x - 3 + 3$$

$$y = 3x$$

Verify with second point: $y = 3(4) = 12$

Form 4: Standard Form

$$Ax + By + C = 0$$

Or:

$$Ax + By = C$$

Where A, B, C are integers with no common factors.

Convert from slope-intercept to standard:

$$y = 2x - 3 \Rightarrow 2x - y - 3 = 0$$

Finding Intercepts

Y-Intercept

The y-intercept is where the line crosses the y-axis (when $x = 0$).

To find: Set $x = 0$ and solve for y.

> 📋 **Example**
> **Find y-intercept of $2x + 3y = 6$:**
>
> Set $x = 0$:
>
> $$2(0) + 3y = 6$$
>
> $$3y = 6$$
>
> $$y = 2$$
>
> **Y-intercept:** $(0, 2)$

X-Intercept

The x-intercept is where the line crosses the x-axis (when $y = 0$).

To find: Set $y = 0$ and solve for x.

📋 **Example**

Find x-intercept of $2x + 3y = 6$:

Set $y = 0$:

$$2x + 3(0) = 6$$

$$2x = 6$$

$$x = 3$$

X-intercept: $(3, 0)$

🧠 **Remember**

To find intercepts, substitute ZERO for the other variable:

- Y-intercept: Set $x = 0$, solve for y
- X-intercept: Set $y = 0$, solve for x

These are always single points (unless the line doesn't cross that axis, which is rare for linear functions).

Properties of Linear Functions

Parallel Lines

Two lines are **parallel** if they have the same slope and different y-intercepts.

$$y = mx + c_1 \text{ and } y = mx + c_2 \text{ (where } c_1 \neq c_2)$$

Parallel lines never intersect.

📋 **Example**

Lines $y = 2x + 3$ and $y = 2x - 5$ are parallel:

Both have slope $m = 2$, but different y-intercepts (3 and -5).

📋 **Example**

Find a line parallel to $3x + 2y = 7$ passing through $(1, 4)$:

Step 1: Find the slope of $3x + 2y = 7$

$$2y = -3x + 7$$

$$y = -\frac{3}{2}x + \frac{7}{2}$$

Slope: $m = -\frac{3}{2}$

Step 2: Parallel line has same slope: $m = -\frac{3}{2}$

Step 3: Use point-slope form with $(1, 4)$

$$y - 4 = -\frac{3}{2}(x - 1)$$

$$y - 4 = -\frac{3}{2}x + \frac{3}{2}$$

$$y = -\frac{3}{2}x + \frac{11}{2}$$

Perpendicular Lines

Two lines are **perpendicular** if their slopes are negative reciprocals of each other.

If line 1 has slope $m1$ and line 2 has slope $m2$:

$$m_1 \times m_2 = -1$$

Or:

$$m_2 = -\frac{1}{m_1}$$

Perpendicular lines meet at a 90° angle.

Examples:

- Slope 2 and slope $-\frac{1}{2}$ are perpendicular (because $2 \times (-\frac{1}{2}) = -1$)
- Slope 3 and slope $-\frac{1}{3}$ are perpendicular
- Slope $\frac{3}{4}$ and slope $-\frac{4}{3}$ are perpendicular

📋 **Example**

Find a line perpendicular to $y = 2x + 5$ passing through $(3, -1)$:

Step 1: Slope of given line: $m_1 = 2$

Step 2: Perpendicular slope: $m_2 = -\frac{1}{2}$ (negative reciprocal)

Step 3: Use point-slope form with $(3, -1)$

$$y - (-1) = -\frac{1}{2}(x - 3)$$

$$y + 1 = -\frac{1}{2}x + \frac{3}{2}$$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

Length and Midpoint of Line Segments

Distance (Length) Formula

For two points $(x1, y1)$ and $(x2, y2)$, the distance between them is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This comes from the Pythagorean theorem.

📋 **Example**

Find the distance between $(1, 2)$ and $(4, 6)$:

$$d = \sqrt{(4 - 1)^2 + (6 - 2)^2}$$

$$= \sqrt{3^2 + 4^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25}$$

$$= 5 \text{ units}$$

Midpoint Formula

The midpoint of a line segment between $(x1, y1)$ and $(x2, y2)$ is:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Just average the x-coordinates and average the y-coordinates.

📋 **Example**

Find the midpoint between $(2, 3)$ and $(8, 7)$:

$$\text{Midpoint} = \left(\frac{2+8}{2}, \frac{3+7}{2}\right) = \left(\frac{10}{2}, \frac{10}{2}\right) = (5, 5)$$

Check: Distance from $(2, 3)$ to $(5, 5)$ is $\sqrt{9+4} = \sqrt{13}$
Distance from $(5, 5)$ to $(8, 7)$ is $\sqrt{9+4} = \sqrt{13}$
Both equal, so $(5, 5)$ is truly the midpoint.

Graphing Linear Functions

Sketching from Slope-Intercept Form

Given $y = mx + c$:

Step 1: Plot the y-intercept $(0, c)$

Step 2: Use slope $m = \frac{\text{rise}}{\text{run}}$ to find more points:

- If $m = 2 = \frac{2}{1}$: Go right 1, up 2
- If $m = -3 = \frac{-3}{1}$: Go right 1, down 3
- If $m = \frac{2}{3}$: Go right 3, up 2

Step 3: Plot at least 3 points and draw the line through them

📋 **Example**

Sketch $y = -\frac{1}{2}x + 3$**:**

- **Y-intercept:** $(0, 3)$, start here
- **Slope:** $-\frac{1}{2} = \frac{-1}{2}$ (right 2, down 1)
- From $(0, 3)$: go right 2, down 1 ' $(2, 2)$
- From $(2, 2)$: go right 2, down 1 ' $(4, 1)$
- **Alternative direction** (left 2, up 1):
- From $(0, 3)$: go left 2, up 1 ' $(-2, 4)$

Solving Systems Graphically

When you have two linear equations, the **solution** is the point where the lines intersect.

📋 **Example**

Solve graphically:

$$y = 2x - 1$$

$$y = -x + 5$$

Step 1: Graph both lines

- Line 1: y-intercept $(0, -1)$, slope 2
- Line 2: y-intercept $(0, 5)$, slope -1

Step 2: Find intersection point

- From the graph: they intersect at $(2, 3)$

Step 3: Verify

- Line 1: $y = 2(2) - 1 = 3$
- Line 2: $y = -(2) + 5 = 3$

Solution: $(2, 3)$