

Matrices & Basic Operations

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Matrices organise numbers into rows and columns so that related calculations can be handled together. The position of each entry matters, which is why matrix operations have stricter rules than ordinary arithmetic.

In the CSEC optional Vectors and Matrices question, you may need to add, subtract, multiply, or interpret matrices. Always check dimensions first. This tells you whether the operation is allowed and what size the answer should be.

What is a Matrix?

A **matrix** is a rectangular array of numbers organized in **rows** (horizontal) and **columns** (vertical).

Example 2×3 Matrix (2 rows, 3 columns):

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Notation: We write $m \times n$ for m rows and n columns.

Why Matrices?

The point of a matrix is organisation. Instead of writing many separate values, you place them in a structure where position carries meaning.

Matrices are used to organize and manipulate data:

- **Spreadsheets:** Rows = records, columns = attributes
- **Image Processing:** Pixel colors stored in matrices
- **Network Analysis:** Connection patterns between nodes
- **Economic Models:** Input-output tables

Matrix Elements

Matrix entries are named by row first, then column. This row-column order is important because reversing it points to a different position.

Elements are referred to by position: a_{ij} means row i , column j

For $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$:

- $a_{11} = 1$ (row 1, column 1)
- $a_{12} = 2$ (row 1, column 2)
- $a_{23} = 6$ (row 2, column 3)

Special Matrix Types

Special matrices behave like familiar numbers in matrix arithmetic. The zero matrix acts like 0 for addition, and the identity matrix acts like 1 for multiplication.

Square Matrix: Same number of rows and columns

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad (2 \times 2)$$

Identity Matrix I : Square matrix with 1s on diagonal, 0s elsewhere

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zero Matrix: All elements are 0

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrix Addition and Subtraction

Rules

Addition and subtraction are position-by-position operations. This is why the matrices must have the same dimensions.

Can only add/subtract matrices of the same dimensions($m \times n$)

Addition: Add corresponding elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

Subtraction: Subtract corresponding elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$$

Examples

Example

$$\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5+1 & 3+(-1) \\ 2+3 & 4+2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 6 \end{pmatrix}$$

Example

$$\begin{pmatrix} 7 & 4 \\ 2 & 9 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 7-3 & 4-1 \\ 2-1 & 9-5 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 4 \end{pmatrix}$$

Remember

You can only add or subtract matrices with **same dimensions**. A 2×3 matrix cannot be added to a 3×3 matrix.

Scalar Multiplication of Matrices

Definition

Scalar multiplication changes the size of every entry by the same factor. It does not change the dimensions of the matrix.

Multiply every element by the scalar

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Examples

Example

$$3 \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 12 & 15 \end{pmatrix}$$

Example

$$-2 \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ -4 & -8 \end{pmatrix}$$

Properties

- 1. **Associative:** $(rs)A = r(sA)$
- 2. **Distributive:** $r(A + B) = rA + rB$
- 3. **Distributive:** $(r + s)A = rA + sA$

Matrix Multiplication

Why Matrix Multiplication is Different

Matrix multiplication combines rows from the first matrix with columns from the second. Because rows and columns play different roles, changing the order can change the result.

Key Idea: Matrix multiplication is NOT element-by-element. It represents **combining transformations** or **composing operations**.

Rule: Can only multiply if: (columns of first matrix) = (rows of second matrix)

$$(m \times n) \times (n \times p) = (m \times p)$$

How to Multiply: Step-by-Step

Work systematically across the answer matrix. Each blank entry has its own row-column calculation, so label positions if the matrix is large.

For matrices A and B to get product AB :

Each element in the product = (row of A) · (column of B)

$$AB_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$$

2×2 Matrix Multiplication Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = ?$$

Position (1,1): (Row 1 of first) · (Column 1 of second)

$$= 1(5) + 2(7) = 5 + 14 = 19$$

Position (1,2): (Row 1 of first) · (Column 2 of second)

$$= 1(6) + 2(8) = 6 + 16 = 22$$

Position (2,1): (Row 2 of first) · (Column 1 of second)

$$= 3(5) + 4(7) = 15 + 28 = 43$$

Position (2,2): (Row 2 of first) · (Column 2 of second)

$$= 3(6) + 4(8) = 18 + 32 = 50$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Remember

Matrix multiplication is **NOT commutative**: $AB \neq BA$. Order matters!

2×3 times 3×2 Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$

Result is 2×2:

$$\text{Position (1,1): } 1(7) + 2(9) + 3(11) = 7 + 18 + 33 = 58$$

$$\text{Position (1,2): } 1(8) + 2(10) + 3(12) = 8 + 20 + 36 = 64$$

$$\text{Position (2,1): } 4(7) + 5(9) + 6(11) = 28 + 45 + 66 = 139$$

$$\text{Position (2,2): } 4(8) + 5(10) + 6(12) = 32 + 50 + 72 = 154$$

$$= \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}$$

Exam Tip

Always check dimensions first:

- Can I multiply? (columns of first = rows of second)
- What size is result? (rows of first × columns of second)
- Multiply systematically (row × column for each position)

Multiplying Matrix by Column Vector

Multiplying a matrix by a column vector is how transformations act on points. The input point goes in as a vector, and the output vector gives the transformed point.

Key Use: Transform a point or apply an operation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Example

Rotation of point $(1, 2)$ by 90° counterclockwise uses matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0(1) + (-1)(2) \\ 1(1) + 0(2) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Point $(1, 2)$ rotated 90° becomes $(-2, 1)$.

Determinants

What is a Determinant?

For a 2×2

matrix, the determinant is a single number that tells whether the matrix can be reversed. A determinant of zero means information has collapsed and no inverse exists.

The **determinant** of a square matrix is a special number that:

- Tells if a matrix is invertible
- Represents scaling factor in transformations
- Used in solving systems of equations

Notation: $\det(A)$ or $|A|$

2x2 Determinant

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Mnemonic: Multiply diagonal down-left, subtract diagonal down-right

2x2 Determinant Examples

Example

$$\det \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = 3(4) - 2(1) = 12 - 2 = 10$$

 **Example**

$$\det \begin{pmatrix} 5 & 10 \\ 2 & 4 \end{pmatrix} = 5(4) - 10(2) = 20 - 20 = 0$$

Determinant 0 means the matrix is **singular** (not invertible).

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