Determinants, Inverses & Matrix Transformations

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Matrices give a compact way to perform several calculations at once. In the optional CSEC section on Vectors and Matrices, you may be asked to use matrices to undo operations, solve simultaneous equations, or transform points on a plane.

The key idea is that matrix work is highly procedural, but it still needs interpretation. A determinant tells you whether an inverse exists; an inverse matrix reverses the effect of another matrix; a transformation matrix moves points in a predictable way. State what the matrix is doing before you start multiplying.

What is an Inverse?

An inverse matrix reverses the effect of a matrix multiplication, similar to how division reverses multiplication with ordinary numbers.

The **inverse** of matrix [Math: A] is matrix [Math: A^{-1}] such that:

AA^{-1} = A^{-1}A = I

Think of it like division: [Math: A^{-1}] "undoes" what [Math: A] does.

When Does an Inverse Exist?

Only when [Math: \det(A) \neq 0] (called a non-singular matrix)

Formula for 2×2 Inverse

The formula swaps the diagonal entries, changes the signs of the other diagonal, and divides by the determinant. The determinant check comes first because division by zero is impossible.

For [Math: A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}]:

A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}

Steps:

- 1. Calculate determinant: [Math: ad - bc]
- 2. Swap [Math: a] and [Math: d]
- 3. Negate [Math: b] and [Math: c]
- 4. Divide by determinant

2×2 Inverse Example

Find the inverse of [Math: A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}]

Step 1: Find determinant

\det(A) = 3(4) - 1(2) = 12 - 2 = 10

Step 2: Apply formula

A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{pmatrix}

Step 3: Verify (multiply [Math: A \cdot A^{-1}])

\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{pmatrix}

Position (1,1): [Math: 3(0.4) + 1(-0.2) = 1.2 - 0.2 = 1]

Position (1,2): [Math: 3(-0.1) + 1(0.3) = -0.3 + 0.3 = 0]

Position (2,1): [Math: 2(0.4) + 4(-0.2) = 0.8 - 0.8 = 0]

Position (2,2): [Math: 2(-0.1) + 4(0.3) = -0.2 + 1.2 = 1]

Result: [Math: I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]

> 📝 **Exam Tip**
> Always **verify** your inverse by multiplying [Math: A \cdot A^{-1}] to check you get the identity matrix.

No Inverse When Determinant = 0

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

$$\det(A) = 2(2) - 4(1) = 4 - 4 = 0$$

Since determinant is 0, this matrix has **no inverse**. (Its columns are proportional—row 1 is exactly 2× row 2.)

Part 10: Solving Systems Using Matrices

Converting a System to Matrix Form

Matrix form separates coefficients, unknowns, and constants. This makes a pair of simultaneous equations look like one compact equation: $AX = B$.

System:

$$\begin{aligned} 2x + 3y &= 8 \\ x - y &= 1 \end{aligned}$$

Matrix Form: $AX = B$

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

Where:

- A = coefficient matrix
- X = variable column vector
- B = constants column vector

Solving Using Matrix Inverse

If $AX = B$, multiplying by A^{-1} undoes the coefficient matrix. The result is the unknown vector X.

If $AX = B$, then:

$$X = A^{-1}B$$

(Multiply both sides by A^{-1} on the left)

Complete Example

Problem: Solve $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

Step 1: Find $\det(A)$

$$\det \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = 2(-1) - 3(1) = -2 - 3 = -5$$

Step 2: Find A^{-1}

$$A^{-1} = \frac{1}{-5} \begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 \\ 0.2 & -0.4 \end{pmatrix}$$

Step 3: Multiply $A^{-1}B$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 \\ 0.2 & -0.4 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$$x = 0.2(8) + 0.6(1) = 1.6 + 0.6 = 2.2$$

$$y = 0.2(8) - 0.4(1) = 1.6 - 0.4 = 1.2$$

Answer: $x = 2.2$, $y = 1.2$

Check: $2(2.2) + 3(1.2) = 4.4 + 3.6 = 8$ and $2.2 - 1.2 = 1$

> 🧠 **Remember**
> Matrix method works when $\det(A) \neq 0$. If determinant is 0, the system either has no solution or infinitely many solutions.

Part 11: Transformation Matrices

What Transformations Do Matrices Represent?

Transformation matrices describe movement on the coordinate plane. Each point is written as a column vector, multiplied by the matrix, and changed into its image.

Matrices can represent:

- **Rotation:** Turn a shape around origin
- **Reflection:** Mirror a shape
- **Scaling:** Enlarge or reduce
- **Shear:** Skew a shape

How it Works: To transform point [Math: (x, y)], multiply by transformation matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation Matrix

Rotate counterclockwise by angle [Math: \theta]:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

📋 **Example**

Rotate 90° counterclockwise: [Math: \cos(90\u00b0) = 0], [Math: \sin(90\u00b0) = 1]

[MathBlock]
R(90\u00b0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
[/MathBlock]

Transform [Math: (1, 0)]:

[MathBlock]
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =
\begin{pmatrix} 0 \\ 1 \end{pmatrix}
[/MathBlock]

Point [Math: (1, 0)] '[Math: (0, 1)] (rotated 90°)

Reflection Matrix

Reflect across x-axis:

$$R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reflect across y-axis:

$$R_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Reflect across line [Math: y = x]:

$$R_{y=x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Scaling Matrix

Scale by factor [Math: k] (same in both directions):

$$S(k) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Scale different in each direction (x by [Math: a], y by [Math: b]):

$$S(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

> 📋 **Example**
> Double the size of point [Math: (2, 3)]:
>
> [MathBlock]
> \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} =
> \begin{pmatrix} 4 \\ 6 \end{pmatrix}
> [/MathBlock]

Composite Transformations

Composite transformations are performed in sequence. In matrix form, the matrix nearest the point acts first, so the written order must be handled carefully.

Apply multiple transformations by **multiplying matrices**

To apply transformation [Math: T2] after [Math: T1]:

\text{Result} = T_2 \cdot T_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix}

(Work right to left: apply [Math: T1] first, then [Math: T2])

[Code: plaintext]

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ANIMATION: Composite transformations
- Show original triangle
- Apply first transformation (e.g., rotation)
- Show intermediate result
- Apply second transformation (e.g., scaling)
- Show final result
- Show that multiplying matrices in order gives same result
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Part 12: Non-Commutativity of Matrix Multiplication

Matrix Multiplication is Not Commutative

Critical Rule: [Math: AB \neq BA] in general

Unlike multiplication of numbers, matrix multiplication depends on **order**.

📋 **Example**

[MathBlock]
A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
[/MathBlock]

[MathBlock]
AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}
[/MathBlock]

[MathBlock]
BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}
[/MathBlock]

Notice: [Math: AB \neq BA]

Why It Matters: In transformations, the order you apply them matters completely. Rotating then scaling gives a different result than scaling then rotating.

📝 **Exam Tip**

When composing two transformations represented by matrices [Math: T1] and [Math: T2]:

- To apply [Math: T1] **first**, then [Math: T2]: multiply as [Math: T2 \cdot T1] (right to left)
- **Never assume** [Math: T1 \cdot T2 = T2 \cdot T1]—they're different transformations

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Summary: Key Concepts

<table>
<tr><th colspan="2">Concept</th><th colspan="2">Definition</th><th>Key Formula</th></tr>
<tr><td>Vector</td><td>Object with magnitude and direction</td><td>[Math: \</td><td>\vec{v}\</td><td>= \sqrt{x^2 + y^2}]</td></tr>
<tr><td colspan="2">Vector Addition</td><td colspan="2">Combine vectors head-to-tail</td><td>[Math: \vec{u} + \vec{v}] component-wise</td></tr>
<tr><td colspan="2">Matrix</td><td colspan="2">Rectangular array of numbers</td><td>[Math: A = (a_{ij})] rows × columns</td></tr>
<tr><td colspan="2">Matrix Product</td><td colspan="2">Row × Column combination</td><td>[Math: AB_{ij} = \text{row } i \cdot \text{col } j]</td></tr>
<tr><td colspan="2">2×2 Determinant</td><td colspan="2">Scalar value of matrix</td><td>[Math: \det(A) = ad - bc]</td></tr>
</table>

2×2 Inverse	Undoes matrix multiplication	[Math: A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}]
Rotation Matrix	Rotates by angle [Math: \theta]	[Math: \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}]