

Relations & Functions

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Relations and functions describe how inputs are paired with outputs. Every function is a relation, but not every relation is a function, because a function gives each input exactly one output.

In CSEC, this topic often asks you to move between representations: ordered pairs, mapping diagrams, tables, equations, and graphs. Keep track of the domain, codomain, and range, because those words describe the input choices, possible targets, and actual outputs.

A Relation Is a Connection

A **relation** is simply a connection between two sets of things. It pairs up elements from one set with elements from another set.

Real-world examples:

- Each student paired with their test score
- Each city paired with its temperature
- Each book paired with its author
- Each date paired with the day's closing stock price

Sets and Notation

Before we formalize relations, understand the vocabulary:

- **Set:** A collection of objects (written in curly braces)
- Example: $A = \{1, 2, 3, 4, 5\}$ (the set of single digits 1-5)
- Example: $B = \{a, e, i, o, u\}$ (vowels)
- **Ordered pair:** Two elements in a specific order, written as (x, y)
- $(3, 6)$ is different from $(6, 3)$
- First element: input
- Second element: output
- **Cartesian product:** All possible ordered pairs from two sets
- If $A = \{1, 2\}$ and $B = \{a, b\}$
- Then $A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$ (4 pairs total)

Defining a Relation Formally

A **relation from set A to set B** is a subset of the Cartesian product $A \times B$.

In simpler terms: Pick some (but not necessarily all) ordered pairs from $A \times B$.

Example

Let $A = \{1, 2, 3\}$ (ages) and $B = \{68, 72, 76\}$ (heights in inches)

The relation " R : person of age x has height y " might be:

[MathBlock]

$$R = \{(1, 68), (2, 72), (3, 76)\}$$

[/MathBlock]

Or it could be any other subset, like $\{(1, 72), (3, 76)\}$ (not all pairs need to be included).

Cartesian product $A \times B$ has 9 possible pairs total.

Our relation R uses only 3 of them.

Key Properties of Relations

Domain — The Input Set

The **domain** is the set of all first elements (inputs).

Example

For relation $R = \{(2, 5), (3, 7), (4, 9), (2, 11)\}$:

Domain: $\{2, 3, 4\}$ (list each once, even if it appears multiple times)

Note: 2 is in the domain only once, even though it appears twice in the relation.

Codomain — The Target Set

The **codomain** is the set we're pairing TO (the "target" set we might use, whether we actually use all elements or not).

Range (or Image) — The Output Set

The **range** is the set of all second elements (outputs) that actually appear.

Example

For relation [Math: $R = \{(2, 5), (3, 7), (4, 9), (2, 11)\}$]:

If codomain is [Math: $\{5, 6, 7, 8, 9, 10, 11\}$]:

- **Codomain** (target): [Math: $\{5, 6, 7, 8, 9, 10, 11\}$] (7 elements, some unused)
- **Range** (actually used): [Math: $\{5, 7, 9, 11\}$] (4 elements, only the actual outputs)

Key difference: Codomain is "available to use," Range is "actually used."

Types of Relations

One-to-One (Injective)

Each input connects to exactly one output, and no two inputs connect to the same output.

Example: Student 'Student ID (each student has one unique ID)

<Mermaid id="rel-one-one" title="One-to-one: each input 'unique output" chart={`flowchart LR

1((1)) --> a((a))

2((2)) --> b((b))

3((3)) --> c((c))`} />

Many-to-One

Multiple inputs can connect to the same output.

Example: City 'Country (many cities in one country)

<Mermaid id="rel-many-one" title="Many-to-one: 1a, 2a, 3b" chart={`flowchart LR

1((1)) --> a((a))

2((2)) --> a

3((3)) --> b((b))`} />

One-to-Many

One input connects to multiple outputs.

Example: Person 'Hobbies (one person has many hobbies)

```
<Mermaid id="rel-one-many" title="One-to-many: 1a, 1b, 2c" chart={`flowchart LR
1((1)) --> a((a))
1 --> b((b))
2((2)) --> c((c))` />
```

Many-to-Many

Inputs and outputs can have multiple connections.

Example: Students 'Subjects (many students in many subjects)

```
<Mermaid id="rel-many-many" title="Many-to-many: 1a, 1b, 2a, 2b" chart={`flowchart LR
1((1)) --> a((a))
1 --> b((b))
2((2)) --> a
2 --> b` />
```

Remember

Domain: Set of all FIRST elements (inputs)

Range: Set of all SECOND elements (outputs actually used)

Codomain: Target set (all possible outputs available)

Domain and Range are determined by the actual relation.

Codomain is usually stated separately.

Part 2: Functions — Special Relations

What Makes a Function Special?

A **function** is a special type of relation with ONE strict rule:

Each input must have EXACTLY ONE output.

This means:

- Input 3 connects to output 9 (good)
- Input 5 connects to output 25 (good)
- Input 3 connects to outputs 9 AND 10 (bad! not a function)
- Input 5 connects to outputs 25 AND 5 (bad! not a function)

However: It's okay if two different inputs map to the same output (many-to-one is fine for functions).

Function Notation

When we have a function, we use special notation to show the rule.

Standard Notation

$f: A \to B$

Read as: "Function [Math: f] from set [Math: A] to set [Math: B]"

- [Math: A] = domain (the inputs)
- [Math: B] = codomain (the target outputs)
- The rule defines which output goes with each input

Function Rule Notation

$f(x) = \text{formula}$

This means:

- [Math: f] is the name of the function
- [Math: x] is the input variable
- [Math: $f(x)$] is the output (what you get after applying the rule)

Examples:

- [Math: $f(x) = 2x + 1$] means: "Take the input [Math: x], double it, and add 1"
- [Math: $g(x) = x^2$] means: "Take the input [Math: x] and square it"
- [Math: $h(t) = 5t$] means: "Take the input [Math: t] and multiply by 5"

Evaluating Functions — Finding Outputs

To **evaluate** a function at a specific value, substitute that value in for [Math: x].

Example

Given [Math: $f(x) = 2x + 1$], find [Math: $f(3)$]:

Step 1: Substitute 3 for every [Math: x]

$$f(3) = 2(3) + 1$$

Step 2: Calculate

$$f(3) = 6 + 1 = 7$$

Meaning: When input is 3, output is 7. The point [Math: (3, 7)] is on the graph.

Example

Given [Math: $g(x) = x^2 - 2x$], find [Math: $g(-2)$]:

Step 1: Substitute -2 for every [Math: x] (use brackets!)

$$g(-2) = (-2)^2 - 2(-2)$$

Step 2: Calculate carefully

$$g(-2) = 4 + 4 = 8$$

Answer: [Math: $g(-2) = 8$]

Functions vs. Relations — The Key Difference

The **vertical line test** distinguishes functions from non-functions:

Vertical line test rule: If you draw a vertical line at any [Math: x]-value on the graph, it crosses the graph at most ONCE, the relation is a function.

- **Crosses once:** function
- **Crosses twice or more:** not a function
- **Doesn't cross:** that [Math: x]-value isn't in the domain

<JSXGraph id="rel-vlt" title="Vertical line test: $y=x^2$ passes (function); $x=y^2$ fails (not a function)"

boundingbox={[-5, 5, 5, -5]} height={360}

code="board.create('functiongraph',[function(x){return

xx-3;},-3,3},{strokeColor:'#2563eb',strokeWidth:2,name:' $y=x^2-3$ ',withLabel:true});board.create('curve',[function

tt-3;},function(t){return

t;},-2.5,2.5},{strokeColor:'#dc2626',strokeWidth:2,name:' $x=y^2-3$ ',withLabel:true});board.create('line',[[[-1,-5],[-

/>

Example

Is $y^2 = x$ a function?

Rearrange: $y = \pm\sqrt{x}$

This means for each $x > 0$, there are TWO possible y values (positive and negative square root).

Example: If $x = 4$, then $y = 2$ OR $y = -2$ (two outputs for one input).

Answer: NOT a function. Fails the vertical line test.