Relations & Functions

Matthew Williams • Math • May 6, 2026

Relations and functions describe how inputs are paired with outputs. Every function is a relation, but not every relation is a function, because a function gives each input exactly one output.

In CSEC, this topic often asks you to move between representations: ordered pairs, mapping diagrams, tables, equations, and graphs. Keep track of the domain, codomain, and range, because those words describe the input choices, possible targets, and actual outputs.

A Relation Is a Connection

A **relation** is simply a connection between two sets of things. It pairs up elements from one set with elements from another set.

Real-world examples:

- Each student paired with their test score
- Each city paired with its temperature
- Each book paired with its author
- Each date paired with the day's closing stock price

Sets and Notation

Before we formalize relations, understand the vocabulary:

- **Set:** A collection of objects (written in curly braces)
- Example: $A = \{1, 2, 3, 4, 5\}$ (the set of single digits 1-5)
- Example: $B = \{a, e, i, o, u\}$ (vowels)

- **Ordered pair:** Two elements in a specific order, written as (x, y)
- $(3, 6)$ is different from $(6, 3)$
- First element: input
- Second element: output

- **Cartesian product:** All possible ordered pairs from two sets
- If $A = \{1, 2\}$ and $B = \{a, b\}$
- Then $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$ (4 pairs total)

Defining a Relation Formally

A **relation from set A to set B** is a subset of the Cartesian product $A \times B$.

In simpler terms: Pick some (but not necessarily all) ordered pairs from $A \times B$.

📋 **Example**

Let $A = \{1, 2, 3\}$ (ages) and $B = \{68, 72, 76\}$ (heights in inches)

The relation "R: person of age x has height y" might be:

$$R = \{(1, 68), (2, 72), (3, 76)\}$$

Or it could be any other subset, like $\{(1, 72), (3, 76)\}$ (not all pairs need to be included).

Cartesian product $A \times B$ has 9 possible pairs total.

Our relation R uses only 3 of them.

Key Properties of Relations

Domain

The **domain** is the set of all first elements (inputs).

📋 **Example**

For relation $R = \{(2, 5), (3, 7), (4, 9), (2, 11)\}$:

Domain: $\{2, 3, 4\}$ (list each once, even if it appears multiple times)

Note: 2 is in the domain only once, even though it appears twice in the relation.

Codomain

The **codomain** is the set we're pairing TO (the "target" set we might use, whether we actually use all elements or not).

Range (or Image)

The **range** is the set of all second elements (outputs) that actually appear.

📋 **Example**
For relation $R = \{(2,5),(3,7),(4,9),(2,11)\}$:
If codomain is $\{5,6,7,8,9,10,11\}$:

- **Codomain** (target): $\{5,6,7,8,9,10,11\}$ (7 elements, some unused)
- **Range** (actually used): $\{5,7,9,11\}$ (4 elements, only the actual outputs)

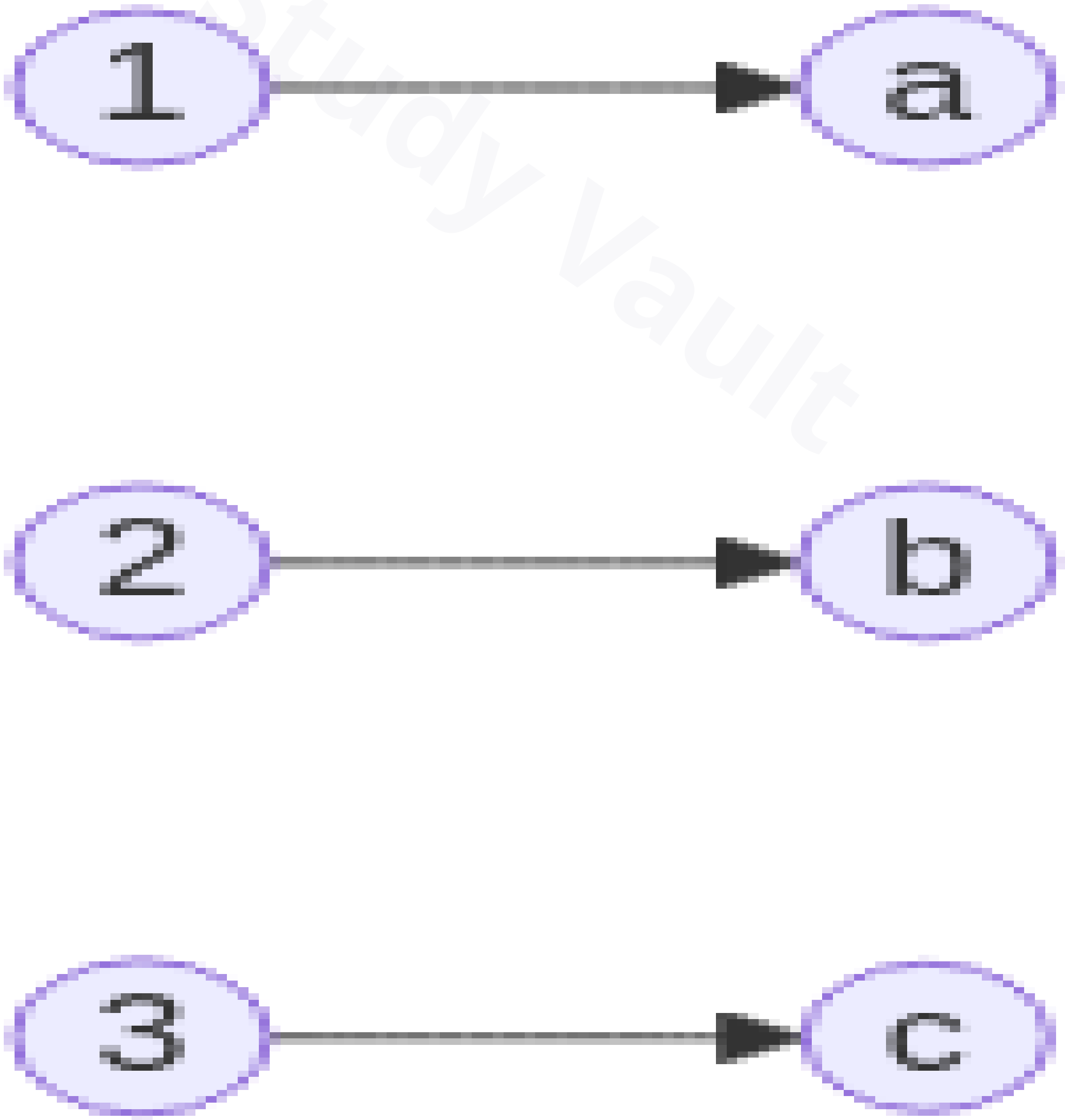
Key difference: Codomain is "available to use," Range is "actually used."

Types of Relations

One-to-One (Injective)

Each input connects to exactly one output, and no two inputs connect to the same output.

Example: Student 'Student ID (each student has one unique ID)

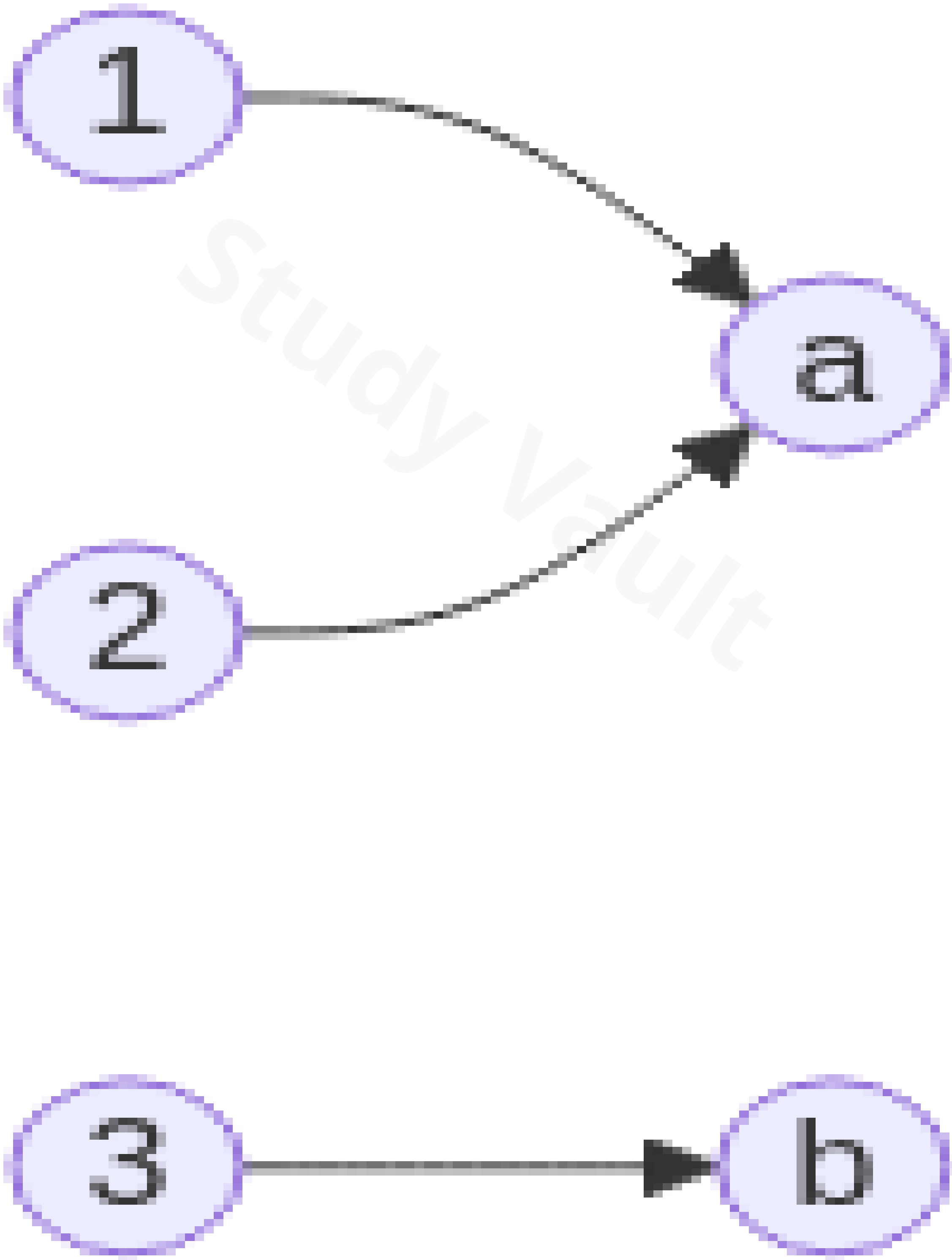


One-to-one: each input 'unique output

Many-to-One

Multiple inputs can connect to the same output.

Example: City ’Country (many cities in one country)

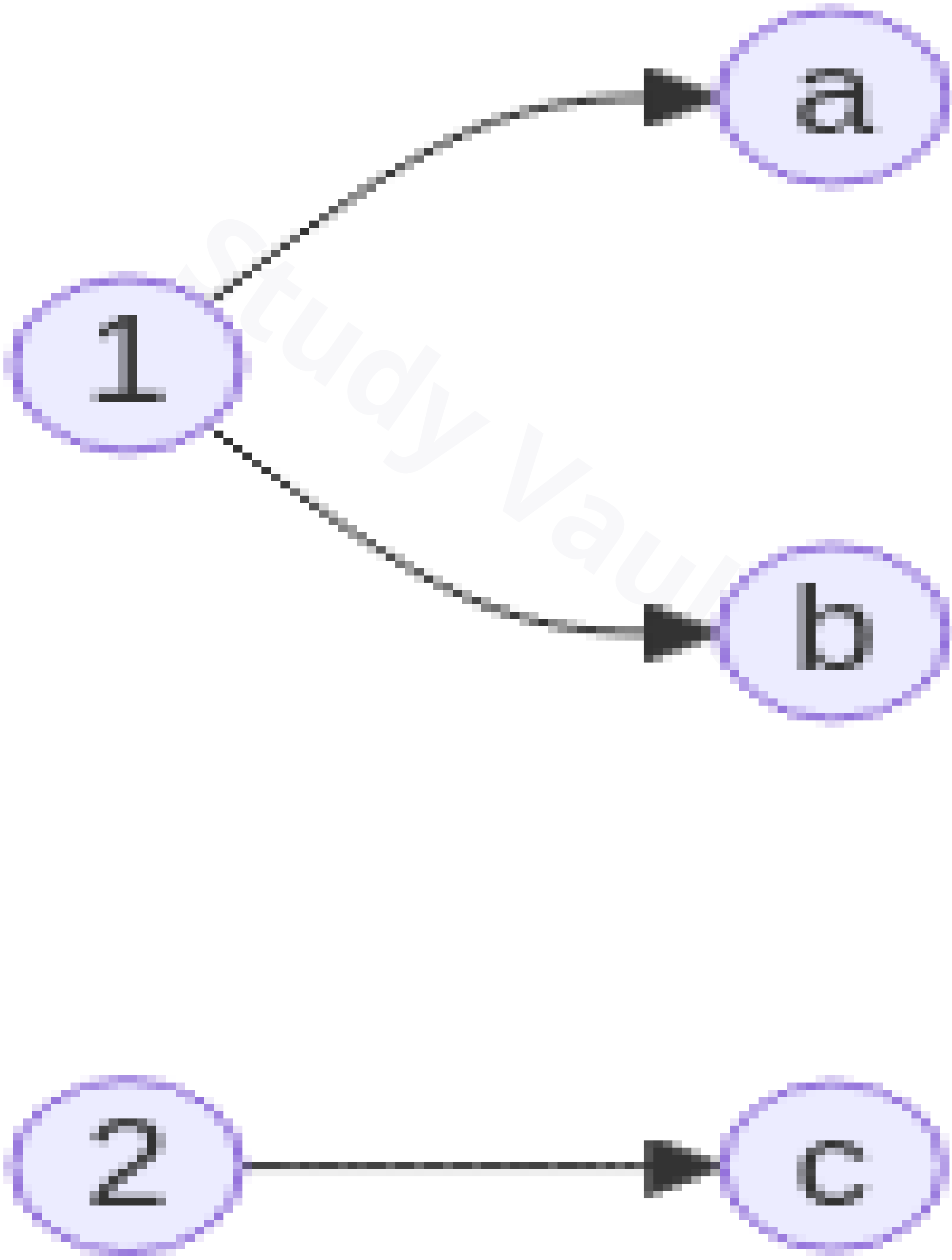


Many-to-one: 1’a, 2’a, 3’b

One-to-Many

One input connects to multiple outputs.

Example: Person ’Hobbies (one person has many hobbies)

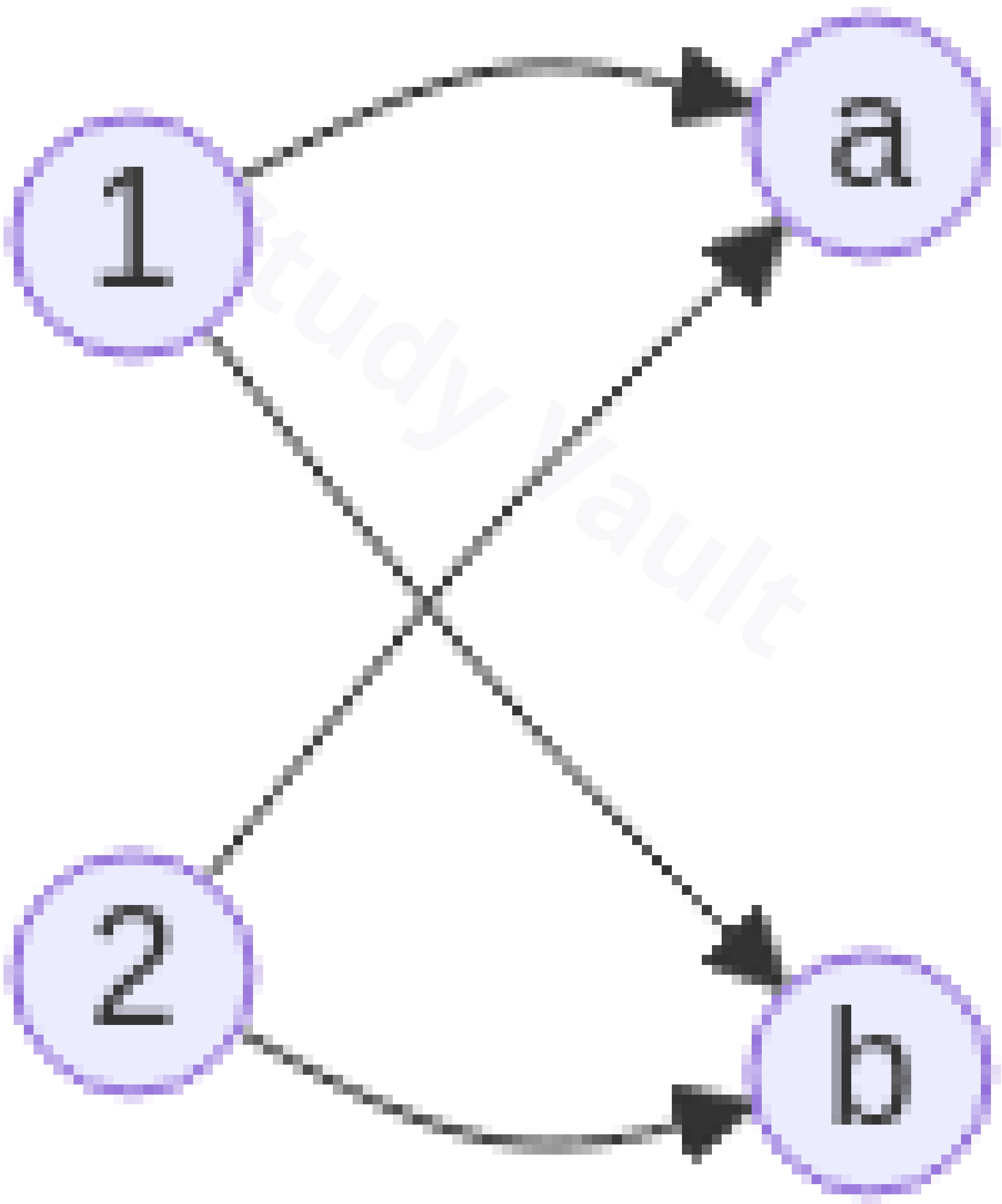


One-to-many: 1’a, 1’b, 2’c

Many-to-Many

Inputs and outputs can have multiple connections.

Example: Students 'Subjects (many students in many subjects)



Many-to-many: 1à, 1b, 2à, 2b

> **Remember**
> **Domain:** Set of all FIRST elements (inputs)
> **Range:** Set of all SECOND elements (outputs actually used)
> **Codomain:** Target set (all possible outputs available)
>
> Domain and Range are determined by the actual relation.
> Codomain is usually stated separately.

Functions

What Makes a Function Special?

A **function** is a special type of relation with ONE strict rule:

Each input must have EXACTLY ONE output.

This means:

- Input 3 connects to output 9 (good)
- Input 5 connects to output 25 (good)
- Input 3 connects to outputs 9 AND 10 (bad! not a function)
- Input 5 connects to outputs 25 AND 5 (bad! not a function)

However: It's okay if two different inputs map to the same output (many-to-one is fine for functions).

Function Notation

When we have a function, we use special notation to show the rule.

Standard Notation

$$f : A \to B$$

Read as: "Function f from set A to set B"

- A = domain (the inputs)
- B = codomain (the target outputs)
- The rule defines which output goes with each input

Function Rule Notation

$$f(x) = \text{formula}$$

This means:

- f is the name of the function
- x is the input variable
- $f(x)$ is the output (what you get after applying the rule)

Examples:

- $f(x) = 2x + 1$ means: "Take the input x, double it, and add 1"
- $g(x) = x^2$ means: "Take the input x and square it"
- $h(t) = 5t$ means: "Take the input t and multiply by 5"

Evaluating Functions

To **evaluate** a function at a specific value, substitute that value in for x.

📋 **Example**

Given $f(x) = 2x + 1$, find $f(3)$:

Step 1: Substitute 3 for every x

$$f(3) = 2(3) + 1$$

Step 2: Calculate

$$f(3) = 6 + 1 = 7$$

Meaning: When input is 3, output is 7. The point $(3, 7)$ is on the graph.

📋 **Example**

Given $g(x) = x^2 - 2x$ **, find** $g(-2)$ **:**

Step 1: Substitute -2 for every x (use brackets!)

$$g(-2) = (-2)^2 - 2(-2)$$

Step 2: Calculate carefully

$$g(-2) = 4 + 4 = 8$$

Answer: $g(-2) = 8$

Functions vs. Relations

The **vertical line test** distinguishes functions from non-functions:

Vertical line test rule: If you draw a vertical line at any x

-value on the graph, it crosses the graph at most ONCE, the relation is a function.

- **Crosses once:** function
- **Crosses twice or more:** not a function
- **Doesn't cross:** that x -value isn't in the domain

📋 **Example**

Is $y^2 = x$ **a function?**

Rearrange: $y = \pm\sqrt{x}$

This means for each $x > 0$, there are TWO possible y
values (positive and negative square root).

Example: If $x = 4$, then $y = 2$ OR $y = -2$ (two outputs for one input).

Answer: NOT a function. Fails the vertical line test.